Operational Semantics of OCaml

- Present three systems
  - OS_{subst} : substitution model
  - OS_{clo} : closure model
  - OS_{state} : closure model, plus state
- In all systems, start with removing let's and letrec's as follows:
  - let x = e in e' => (fun x -> e')e
  - let rec f = e in e' => (fun f -> e')(rec f = e)
- Here, "rec f = e" is a new expression added just for the operational semantics. Note: here, e must be an abstraction.

OS_{subst}

- Just like OS_{simp}, but with recursion.
- Expressions:
  - constants (not higher order), var, application, abstraction, built-in function calls: $e_1 \oplus e_2$
  - Note: Partial application of built-ins is not allowed. This implies that in an application $e_1 e_2$, e_1 must reduce to a user-defined function.
- Values:
  - constants, (closed) abstractions
- Judgments:
  - $e \Downarrow v$ (e closed)

Axioms

- (Const) $$\frac{}{k \Downarrow k}$$
- (Abstr) $$\frac{}{fun\ x \rightarrow e \Downarrow fun\ x \rightarrow e}$$
- (Rec) $$\frac{}{rec\ f = e \Downarrow e\,[rec\ f = e\ /\ f]}$$

Remember: e is an abstraction

Rules of inference (Same as OS_{simp})

- (δ) $$\frac{e_1 \Downarrow v_1 \qquad e_2 \Downarrow v_2 \qquad v = v_1 \oplus v_2}{e_1 \oplus e_2 \Downarrow v}$$

- (if-true) $$\frac{e_1 \Downarrow \mathit{true} \qquad e_2 \Downarrow v}{\mathit{if}\ e_1\ \mathit{then}\ e_2\ \mathit{else}\ e_3 \Downarrow v}$$

- (if-false) $$\frac{e_1 \Downarrow \mathit{false} \qquad e_3 \Downarrow v}{\mathit{if}\ e_1\ \mathit{then}\ e_2\ \mathit{else}\ e_3 \Downarrow v}$$

Never gave these in OS_{simp}

Continued

- (application)

$$\frac{e_1 \Downarrow fun\ x \rightarrow e \quad e_2 \Downarrow v \quad e[v/x] \Downarrow v'}{e_1 e_2 \Downarrow v'}$$

Example

let rec f = fun n -> if n=1 then 1 else n*f(n-1) in f(2) $\Downarrow$ 2

OS_{clo}

- Same expressions; same translation of let and letrec.
- Definition:
  - Environments (notated η): map from variable to value
  - Closures = Expression × Env (notated $\langle e, \eta \rangle$)

must be an abstraction

must contain a value for every free variable in expression

  - Note that there is a circularity here: Env's contain closures and closures contain env's. We'll just keep this informal.

Axioms of OS_{clo}

- (Const) $$\frac{}{\eta, k \Downarrow k}$$ (Var) $$\frac{}{\eta, x \Downarrow \eta(x)}$$
- (Abstr) $$\frac{}{\eta, fun\ x \rightarrow e \Downarrow \langle fun\ x \rightarrow e, \eta \rangle}$$
- (Rec) $$\frac{}{\eta, rec\ f = e \Downarrow \langle e, \eta' \rangle}$$ $$\text{where } \eta' = \eta[f \rightarrow \langle e, \eta' \rangle]$$

Again, be informal about this.
Also, again note that e is an abstraction

Rules of inference

- (δ) $$\frac{\eta, e_1 \Downarrow v_1 \quad \eta, e_2 \Downarrow v_2 \quad v = v_1 \oplus v_2}{\eta, e_1 \oplus e_2 \Downarrow v}$$

- (App)

$$\frac{\eta, e_1 \Downarrow \langle fun\ x \rightarrow e, \eta' \rangle \quad \eta, e_2 \Downarrow v \quad \eta'[x \rightarrow v], e \Downarrow v'}{\eta, e_1 e_2 \Downarrow v'}$$

Example

$$\overline{\varnothing, (\text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3)4 \Downarrow 7}$$

Example

$$\frac{\dfrac{}{\emptyset, (\text{fun } x \to (\text{fun } y \to x+y)3) \Downarrow <\text{fun } x \to (\text{fun } y \to x+y)3, \emptyset>} \qquad \dfrac{}{\emptyset, 4 \Downarrow 4}}{\emptyset, (\text{fun } x \to (\text{fun } y \to x+y)3)4 \Downarrow 7} \quad \mathbf{A}$$

Example

$$\frac{}{\mathbf{A} = [x:4],\ (\text{fun } y \rightarrow x+y)3 \Downarrow 7}$$

$$\frac{\dfrac{}{\emptyset,\ (\text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3) \Downarrow \langle \text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3, \emptyset \rangle} \qquad \dfrac{}{\emptyset,\ 4 \Downarrow 4} \qquad \mathbf{A}}{\emptyset,\ (\text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3)4 \Downarrow 7}$$

Example

$$\frac{\overline{[x{:}4],\ (\text{fun } y \rightarrow x{+}y) \Downarrow \langle \text{fun } y{\rightarrow}x{+}y\ ,[x{:}4]\rangle} \qquad \overline{[x{:}4],\ 3 \Downarrow 3} \qquad \mathbf{B}}{\mathbf{A} = [x{:}4],\ (\text{fun } y \rightarrow x{+}y)3 \Downarrow 7}$$

$$\frac{\overline{\varnothing,\ (\text{fun } x \rightarrow (\text{fun } y \rightarrow x{+}y)3) \Downarrow \langle \text{fun } x \rightarrow (\text{fun } y \rightarrow x{+}y)3, \varnothing\rangle} \qquad \overline{\varnothing,\ 4 \Downarrow 4} \qquad \mathbf{A}}{\varnothing,\ (\text{fun } x \rightarrow (\text{fun } y \rightarrow x{+}y)3)4 \Downarrow 7}$$

Example

$$\frac{\overline{[x:4,y:3],\ x \Downarrow 4} \qquad \overline{[x:4,y:3],\ y \Downarrow 3}}{\mathbf{B} = \ [x:4,y:3],\ x+y \Downarrow 7}$$

$$\frac{\overline{[x:4],\ (\text{fun } y \rightarrow x+y) \Downarrow <\text{fun } y \rightarrow x+y\ ,[x:4]>} \qquad \overline{[x:4],\ 3 \Downarrow 3} \qquad \mathbf{B}}{\mathbf{A} = [x:4],\ (\text{fun } y \rightarrow x+y)3 \Downarrow 7}$$

$$\frac{\overline{\varnothing,\ (\text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3) \Downarrow <\text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3, \varnothing>} \qquad \overline{\varnothing,\ 4 \Downarrow 4} \qquad \mathbf{A}}{\varnothing,\ (\text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3)4 \Downarrow 7}$$

OS_{state}

- Add:
  - Locations
    - Notated ℓ, ℓ', ℓ_1, etc
    - Infinite, unstructured set of atoms
  - State
    - Notated σ
    - Map from locations to values
- Values: Constants, locations, closures
- Judgments:

$$\sigma, \eta \vdash e \Downarrow v, \sigma'$$

Axioms and rules of inference

- All rules are same as OS_{clo}, but "thread" state through subcomputation. States are never captured in closures.

- (Const) $\dfrac{}{\sigma,\eta \vdash k \Downarrow k,\sigma}$ (Var) $\dfrac{}{\sigma,\eta \vdash x \Downarrow \eta(x),\sigma}$

- (Abs) $\dfrac{}{\sigma,\eta \vdash fun\ x \to e \Downarrow \langle fun\ x \to e,\eta\rangle,\sigma}$

- (δ) $$\frac{\sigma,\eta \vdash e_1 \Downarrow v_1,\sigma_1 \quad \sigma_1,\eta \vdash e_2 \Downarrow v_2,\sigma_2 \quad v = v_1 \oplus v_2}{\sigma,\eta \vdash e_1 \oplus e_2 \Downarrow v,\sigma_2}$$

Axioms and rules of inference

- New rules for new operators
- (Deref)

$$\frac{\sigma,\eta \vdash e \Downarrow \ell,\sigma' \qquad \ell \textit{ a location} \qquad \sigma'(\ell) = v}{\sigma,\eta \vdash !e \Downarrow v,\sigma'}$$

- (Assign)

$$\frac{\sigma,\eta \vdash e_1 \Downarrow \ell,\sigma' \qquad \sigma',\eta \vdash e_2 \Downarrow v,\sigma''}{\sigma,\eta \vdash e_1 := e_2 \Downarrow (),\sigma''[\ell \to v]}$$

unique value of type unit

- (Ref)

$$\frac{\sigma,\eta \vdash e \Downarrow v,\sigma' \qquad \ell \textit{ a fresh location}}{\sigma,\eta \vdash \textit{ref } e \Downarrow \ell,\sigma'[\ell \to v]}$$

Example

$$\frac{}{\{\ell:0\},\{x:\ell\} \vdash (\text{fun } y \rightarrow (\text{fun } z \rightarrow !x)(y{:=}!y+1))x \Downarrow 1, \{\ell:1\}}$$

Example

$$\dfrac{\dfrac{}{\{\ell{:}0\},\{x{:}\ell\} \vdash (\text{fun } y \rightarrow \ldots) \Downarrow \langle(\text{fun } y \rightarrow \ldots),\ \{x{:}\ell\}\rangle, \{\ell{:}0\}} \qquad \dfrac{}{\{\ell{:}0\},\{x{:}\ell\} \vdash x \Downarrow \ell, \{\ell{:}0\}} \qquad \mathbf{A}}{\{\ell{:}0\},\{x{:}\ell\} \vdash (\text{fun } y \rightarrow (\text{fun } z \rightarrow !x)(y{:=}!y+1))x \Downarrow 1, \{\ell{:}1\}}$$

Example

$$\frac{}{\{\ell{:}0\},\{x{:}\ell,y{:}\ell\} \vdash (\text{fun } z \to !x) \Downarrow \langle \text{fun } z \to !x, \{x{:}\ell,y{:}\ell\}\rangle, \{\ell{:}0\}} \quad \mathbf{B} \quad \mathbf{C}$$

$$A = \{\ell{:}0\},\{x{:}\ell,y{:}\ell\} \vdash (\text{fun } z \to !x)(y{:=}!y+1) \Downarrow 1, \{\ell{:}1\}$$

$$\frac{\dfrac{}{\{\ell{:}0\},\{x{:}\ell\} \vdash (\text{fun } y \to \ldots) \Downarrow \langle (\text{fun } y \to \ldots), \{x{:}\ell\}\rangle, \{\ell{:}0\}} \qquad \dfrac{}{\{\ell{:}0\},\{x{:}\ell\} \vdash x \Downarrow \ell, \{\ell{:}0\}} \qquad \mathbf{A}}{\{\ell{:}0\},\{x{:}\ell\} \vdash (\text{fun } y \to (\text{fun } z \to !x)(y{:=}!y+1))x \Downarrow 1, \{\ell{:}1\}}$$

Example

$$\frac{}{\mathbf{B} = \{\ell{:}0\},\{x{:}\ell, y{:}\ell\} \vdash y{:=}!y+1 \Downarrow (), \{\ell{:}1\}}$$

$$\frac{\dfrac{}{\{\ell{:}0\},\{x{:}\ell, y{:}\ell\} \vdash (\text{fun } z \to !x) \Downarrow \langle \text{fun } z \to !x, \{x{:}\ell, y{:}\ell\}\rangle, \{\ell{:}0\}} \quad \mathbf{B} \quad \mathbf{C}}{\text{A} = \{\ell{:}0\},\{x{:}\ell, y{:}\ell\} \vdash (\text{fun } z \to !x)(y{:=}!y+1) \Downarrow 1, \{\ell{:}1\}}$$

$$\frac{\dfrac{}{\{\ell{:}0\},\{x{:}\ell\} \vdash (\text{fun } y \to \ldots) \Downarrow \langle (\text{fun } y \to \ldots), \{x{:}\ell\}\rangle, \{\ell{:}0\}} \quad \dfrac{}{\{\ell{:}0\},\{x{:}\ell\} \vdash x \Downarrow \ell, \{\ell{:}0\}} \quad \mathbf{A}}{\{\ell{:}0\},\{x{:}\ell\} \vdash (\text{fun } y \to (\text{fun } z \to !x)(y{:=}!y+1))x \Downarrow 1, \{\ell{:}1\}}$$

Example

$$\{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash y \Downarrow \ell,\{\ell{:}0\} \qquad \{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash\ !y+1 \Downarrow 1,\{\ell{:}0\}$$

$$\mathbf{B} = \{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash y{:=}!y+1 \Downarrow (),\{\ell{:}1\}$$

$$\{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash(\text{fun } z \to !x) \Downarrow \langle\text{fun } z\to !x,\{x{:}\ell,y{:}\ell\}\rangle,\{\ell{:}0\} \qquad \mathbf{B} \qquad \mathbf{C}$$

$$A = \{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash(\text{fun } z \to !x)(y{:=}!y+1) \Downarrow 1,\{\ell{:}1\}$$

$$\{\ell{:}0\},\{x{:}\ell\}\vdash(\text{fun } y \to \ldots) \Downarrow \langle(\text{fun } y \to \ldots),\ \{x{:}\ell\}\rangle,\{\ell{:}0\} \qquad \{\ell{:}0\},\{x{:}\ell\}\vdash x \Downarrow \ell,\{\ell{:}0\} \qquad \mathbf{A}$$

$$\{\ell{:}0\},\{x{:}\ell\}\vdash(\text{fun } y \to(\text{fun } z \to !x)(y{:=}!y+1))x \Downarrow 1,\{\ell{:}1\}$$

Example

$$\dfrac{\dfrac{\dfrac{}{\{\ell:0\},\{x:\ell,y:\ell\}\vdash y \Downarrow \ell,\{\ell:0\}} \qquad \{\ell:0\}(\ell)=0}{\{\ell:0\},\{x:\ell,y:\ell\}\vdash !y \Downarrow 0,\{\ell:0\}} \qquad \dfrac{}{\{\ell:0\},\{x:\ell,y:\ell\}\vdash 1 \Downarrow 1,\{\ell:0\}}}{\{\ell:0\},\{x:\ell,y:\ell\}\vdash !y+1 \Downarrow 1,\{\ell:0\}}$$

$$\dfrac{\dfrac{}{\{\ell:0\},\{x:\ell,y:\ell\}\vdash y \Downarrow \ell,\{\ell:0\}} \qquad \{\ell:0\},\{x:\ell,y:\ell\}\vdash !y+1 \Downarrow 1,\{\ell:0\}}{\mathbf{B} = \{\ell:0\},\{x:\ell,y:\ell\}\vdash y:=!y+1 \Downarrow (),\{\ell:1\}}$$

$$\dfrac{\dfrac{}{\{\ell:0\},\{x:\ell,y:\ell\}\vdash (\text{fun } z \to !x) \Downarrow <\text{fun } z \to !x,\{x:\ell,y:\ell\}>,\{\ell:0\}} \qquad \mathbf{B} \qquad \mathbf{C}}{A = \{\ell:0\},\{x:\ell,y:\ell\}\vdash (\text{fun } z \to !x)(y:=!y+1) \Downarrow 1,\{\ell:1\}}$$

$$\dfrac{\dfrac{}{\{\ell:0\},\{x:\ell\}\vdash (\text{fun } y \to \ldots) \Downarrow <(\text{fun } y \to \ldots),\ \{x:\ell\}>,\{\ell:0\}} \qquad \dfrac{}{\{\ell:0\},\{x:\ell\}\vdash x \Downarrow \ell,\{\ell:0\}} \qquad \mathbf{A}}{\{\ell:0\},\{x:\ell\}\vdash (\text{fun } y \to (\text{fun } z \to !x)(y:=!y+1))x \Downarrow 1,\{\ell:1\}}$$

Example

$$\dfrac{\dfrac{}{\{\ell{:}1\},\{x{:}\ell, y{:}\ell, z{:}()\} \vdash x \Downarrow \ell, \{\ell{:}1\}} \qquad \{\ell{:}1\}(\ell)=1}{\mathbf{C} = \{\ell{:}1\},\{x{:}\ell, y{:}\ell, z{:}()\} \vdash !x \Downarrow 1, \{\ell{:}1\}}$$