Operational Semantics of OCaml

- Present three systems
  - OS_{subst} : substitution model
  - OS_{clo} : closure model
  - OS_{state} : closure model, plus state
- In all systems, start with removing let's and letrec's as follows:
  - let x = e in e' => (fun x -> e')e
  - let rec f = e in e' => (fun f -> e')(rec f = e)
- Here, "rec f = e" is a new expression added just for the operational semantics. Note: here, e must be an abstraction.

OS_{subst}

- Just like OS_{simp}, but with recursion.
- Expressions:
  - constants (not higher order), var, application, abstraction, built-in function calls: $e_1 \oplus e_2$
  - Note: Partial application of built-ins is not allowed. This implies that in an application e_1e_2, e_1 must reduce to a user-defined function.
- Values:
  - constants, (closed) abstractions
- Judgments:
  - $e \Downarrow v$ (e closed)

Axioms

- (Const)

$$\frac{}{k \Downarrow k}$$

- (Abstr)

$$\frac{}{fun\ x \to e \Downarrow fun\ x \to e}$$

- (Rec)

$$\frac{}{rec\ f = e \Downarrow e\,[rec\ f = e\ /\ f]}$$

Remember: e is an abstraction

Rules of inference (Same as OS_{simp})

- (δ)

$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2 \quad v = v_1 \oplus v_2}{e_1 \oplus e_2 \Downarrow v}$$

- (if-true)

$$\frac{e_1 \Downarrow true \qquad e_2 \Downarrow v}{if\ e_1\ then\ e_2\ else\ e_3 \Downarrow v}$$

- (if-false)

$$\frac{e_1 \Downarrow false \qquad e_3 \Downarrow v}{if\ e_1\ then\ e_2\ else\ e_3 \Downarrow v}$$

Never gave these in OS_{simp}

Continued

- (application)

$$\frac{e_1 \Downarrow fun\ x \to e \quad e_2 \Downarrow v \quad e[v/x] \Downarrow v'}{e_1e_2 \Downarrow v'}$$

Example

let rec f = fun n -> if n=1 then 1 else n*f(n-1) in f(2) $\Downarrow$ 2

OS_{clo}

- Same expressions; same translation of let and letrec.
- Definition:
  - Environments (notated η): map from variable to value
  - Closures = Expression × Env (notated <e, η>)

must be an abstraction

must contain a value for every free variable in expression

  - Note that there is a circularity here: Env's contain closures and closures contain env's. We'll just keep this informal.

Axioms of OS_{clo}

- (Const) $\frac{}{\eta, k \Downarrow k}$ (Var) $\frac{}{\eta, x \Downarrow \eta(x)}$
- (Abstr) $\frac{}{\eta, \mathit{fun}\ x \to e \Downarrow \langle \mathit{fun}\ x \to e, \eta \rangle}$
- (Rec) $\frac{}{\eta, \mathit{rec}\ f = e \Downarrow \langle e, \eta' \rangle}$

  where $\eta' = \eta[f \to \langle e, \eta' \rangle]$

Again, be informal about this.
Also, again note that e is an abstraction

Rules of inference

- (δ)

$$\frac{\eta, e_1 \Downarrow v_1 \quad \eta, e_2 \Downarrow v_2 \quad v = v_1 \oplus v_2}{\eta, e_1 \oplus e_2 \Downarrow v}$$

- (App)

$$\frac{\eta, e_1 \Downarrow \langle \mathit{fun}\ x \to e, \eta' \rangle \quad \eta, e_2 \Downarrow v \quad \eta'[x \to v], e \Downarrow v'}{\eta, e_1 e_2 \Downarrow v'}$$

Example

$$\frac{}{\varnothing, (\text{fun } x \to (\text{fun } y \to x+y)3)4 \Downarrow 7}$$

Example

$$\frac{\frac{}{\varnothing, (\text{fun } x \to (\text{fun } y \to x+y)3) \Downarrow <\text{fun } x \to (\text{fun } y \to x+y)3, \varnothing>} \quad \frac{}{\varnothing, 4 \Downarrow 4} \quad \mathbf{A}}{\varnothing, (\text{fun } x \to (\text{fun } y \to x+y)3)4 \Downarrow 7}$$

Example

$$\frac{}{\mathbf{A} = [x:4], (\text{fun } y \to x+y)3 \Downarrow 7}$$

$$\frac{\frac{}{\varnothing, (\text{fun } x \to (\text{fun } y \to x+y)3) \Downarrow <\text{fun } x \to (\text{fun } y \to x+y)3, \varnothing>} \quad \frac{}{\varnothing, 4 \Downarrow 4} \quad \mathbf{A}}{\varnothing, (\text{fun } x \to (\text{fun } y \to x+y)3)4 \Downarrow 7}$$

Example

$$\dfrac{\dfrac{}{[x{:}4], (\text{fun } y \rightarrow x{+}y) \Downarrow <\text{fun } y{\rightarrow}x{+}y\ ,[x{:}4]>} \qquad \dfrac{}{[x{:}4],\ 3 \Downarrow 3} \qquad \mathbf{B}}{\mathbf{A =}\ [x{:}4], (\text{fun } y \rightarrow x{+}y)3 \Downarrow 7}$$

$$\dfrac{\dfrac{}{\varnothing, (\text{fun } x \rightarrow (\text{fun } y \rightarrow x{+}y)3) \Downarrow <\text{fun } x \rightarrow (\text{fun } y \rightarrow x{+}y)3, \varnothing>} \qquad \dfrac{}{\varnothing,\ 4 \Downarrow 4} \qquad \mathbf{A}}{\varnothing, (\text{fun } x \rightarrow (\text{fun } y \rightarrow x{+}y)3)4 \Downarrow 7}$$

Example

$$\dfrac{\dfrac{}{[x{:}4,y{:}3],\ x \Downarrow 4} \qquad \dfrac{}{[x{:}4,y{:}3],\ y \Downarrow 3}}{\mathbf{B =}\ [x{:}4,y{:}3],\ x{+}y \Downarrow 7}$$

$$\dfrac{\dfrac{}{[x{:}4], (\text{fun } y \rightarrow x{+}y) \Downarrow <\text{fun } y{\rightarrow}x{+}y\ ,[x{:}4]>} \qquad \dfrac{}{[x{:}4],\ 3 \Downarrow 3} \qquad \mathbf{B}}{\mathbf{A =}\ [x{:}4], (\text{fun } y \rightarrow x{+}y)3 \Downarrow 7}$$

$$\dfrac{\dfrac{}{\varnothing, (\text{fun } x \rightarrow (\text{fun } y \rightarrow x{+}y)3) \Downarrow <\text{fun } x \rightarrow (\text{fun } y \rightarrow x{+}y)3, \varnothing>} \qquad \dfrac{}{\varnothing,\ 4 \Downarrow 4} \qquad \mathbf{A}}{\varnothing, (\text{fun } x \rightarrow (\text{fun } y \rightarrow x{+}y)3)4 \Downarrow 7}$$

OS_{state}

- Add:
  - Locations
    - Notated ℓ, ℓ', ℓ_1, etc
    - Infinite, unstructured set of atoms
  - State
    - Notated σ
    - Map from locations to values
- Values: Constants, locations, closures
- Judgments:

$$\sigma, \eta \vdash e \Downarrow v, \sigma'$$

Axioms and rules of inference

- All rules are same as OS_{clo}, but “thread” state through subcomputation. States are never captured in closures.
- (Const) $\dfrac{}{\sigma, \eta \vdash k \Downarrow k, \sigma}$ (Var) $\dfrac{}{\sigma, \eta \vdash x \Downarrow \eta(x), \sigma}$
- (Abs) $\dfrac{}{\sigma, \eta \vdash \mathit{fun}\ x \rightarrow e \Downarrow \langle \mathit{fun}\ x \rightarrow e, \eta \rangle, \sigma}$
- (δ) $\dfrac{\sigma, \eta \vdash e_1 \Downarrow v_1, \sigma_1 \quad \sigma_1, \eta \vdash e_2 \Downarrow v_2, \sigma_2 \quad v{=}v_1 \oplus v_2}{\sigma, \eta \vdash e_1 \oplus e_2 \Downarrow v, \sigma_2}$

Axioms and rules of inference

- New rules for new operators
- (Deref) $\dfrac{\sigma, \eta \vdash e \Downarrow \ell, \sigma' \quad \ell \textit{ a location} \quad \sigma'(\ell) = v}{\sigma, \eta \vdash !e \Downarrow v, \sigma'}$
- (Assign) $\dfrac{\sigma, \eta \vdash e_1 \Downarrow \ell, \sigma' \quad \sigma', \eta \vdash e_2 \Downarrow v, \sigma''}{\sigma, \eta \vdash e_1 := e_2 \Downarrow (), \sigma''[\ell \rightarrow v]}$

unique value of type unit

- (Ref) $\dfrac{\sigma, \eta \vdash e \Downarrow v, \sigma' \quad \ell \textit{ a fresh location}}{\sigma, \eta \vdash \mathit{ref}\ e \Downarrow \ell, \sigma'[\ell \rightarrow v]}$

Example

$$\dfrac{}{\{\ell{:}0\}, \{x{:}\ell\} \vdash (\text{fun } y \rightarrow (\text{fun } z \rightarrow !x)(y{:=}!y{+}1))x \Downarrow 1, \{\ell{:}1\}}$$

Example

$$\frac{\frac{}{\{\ell{:}0\},\{x{:}\ell\}\vdash(\text{fun } y\to\ldots)\Downarrow<(\text{fun } y\to\ldots),\ \{x{:}\ell\}>,\{\ell{:}0\}}\qquad\frac{}{\{\ell{:}0\},\{x{:}\ell\}\vdash x\Downarrow\ell,\{\ell{:}0\}}\qquad \mathbf{A}}{\{\ell{:}0\},\{x{:}\ell\}\vdash(\text{fun } y\to(\text{fun } z\to!x)(y:=!y+1))x\Downarrow 1,\{\ell{:}1\}}$$

Example

$$\frac{\frac{}{\{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash(\text{fun } z\to!x)\Downarrow<\text{fun } z\to!x,\{x{:}\ell,y{:}\ell\}>,\{\ell{:}0\}}\qquad \mathbf{B}\qquad \mathbf{C}}{A = \{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash(\text{fun } z\to!x)(y:=!y+1)\Downarrow 1,\{\ell{:}1\}}$$

$$\frac{\frac{}{\{\ell{:}0\},\{x{:}\ell\}\vdash(\text{fun } y\to\ldots)\Downarrow<(\text{fun } y\to\ldots),\ \{x{:}\ell\}>,\{\ell{:}0\}}\qquad\frac{}{\{\ell{:}0\},\{x{:}\ell\}\vdash x\Downarrow\ell,\{\ell{:}0\}}\qquad \mathbf{A}}{\{\ell{:}0\},\{x{:}\ell\}\vdash(\text{fun } y\to(\text{fun } z\to!x)(y:=!y+1))x\Downarrow 1,\{\ell{:}1\}}$$

Example

$$\frac{}{\mathbf{B} = \{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash y:=!y+1\Downarrow(),\{\ell{:}1\}}$$

$$\frac{\frac{}{\{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash(\text{fun } z\to!x)\Downarrow<\text{fun } z\to!x,\{x{:}\ell,y{:}\ell\}>,\{\ell{:}0\}}\qquad \mathbf{B}\qquad \mathbf{C}}{A = \{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash(\text{fun } z\to!x)(y:=!y+1)\Downarrow 1,\{\ell{:}1\}}$$

$$\frac{\frac{}{\{\ell{:}0\},\{x{:}\ell\}\vdash(\text{fun } y\to\ldots)\Downarrow<(\text{fun } y\to\ldots),\ \{x{:}\ell\}>,\{\ell{:}0\}}\qquad\frac{}{\{\ell{:}0\},\{x{:}\ell\}\vdash x\Downarrow\ell,\{\ell{:}0\}}\qquad \mathbf{A}}{\{\ell{:}0\},\{x{:}\ell\}\vdash(\text{fun } y\to(\text{fun } z\to!x)(y:=!y+1))x\Downarrow 1,\{\ell{:}1\}}$$

Example

$$\frac{\frac{}{\{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash y\Downarrow\ell,\{\ell{:}0\}}\qquad\frac{}{\{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash !y+1\Downarrow 1,\{\ell{:}0\}}}{\mathbf{B} = \{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash y:=!y+1\Downarrow(),\{\ell{:}1\}}$$

$$\frac{\frac{}{\{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash(\text{fun } z\to!x)\Downarrow<\text{fun } z\to!x,\{x{:}\ell,y{:}\ell\}>,\{\ell{:}0\}}\qquad \mathbf{B}\qquad \mathbf{C}}{A = \{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash(\text{fun } z\to!x)(y:=!y+1)\Downarrow 1,\{\ell{:}1\}}$$

$$\frac{\frac{}{\{\ell{:}0\},\{x{:}\ell\}\vdash(\text{fun } y\to\ldots)\Downarrow<(\text{fun } y\to\ldots),\ \{x{:}\ell\}>,\{\ell{:}0\}}\qquad\frac{}{\{\ell{:}0\},\{x{:}\ell\}\vdash x\Downarrow\ell,\{\ell{:}0\}}\qquad \mathbf{A}}{\{\ell{:}0\},\{x{:}\ell\}\vdash(\text{fun } y\to(\text{fun } z\to!x)(y:=!y+1))x\Downarrow 1,\{\ell{:}1\}}$$

Example

$$\frac{\frac{}{\{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash y\Downarrow\ell,\{\ell{:}0\}}\qquad\frac{\frac{\frac{}{\{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash y\Downarrow\ell,\{\ell{:}0\}}\qquad \{\ell{:}0\}(\ell)=0}{\{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash !y\Downarrow 0,\{\ell{:}0\}}\qquad\frac{}{\{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash 1\Downarrow 1,\{\ell{:}0\}}}{\{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash !y+1\Downarrow 1,\{\ell{:}0\}}}{\mathbf{B} = \{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash y:=!y+1\Downarrow(),\{\ell{:}1\}}$$

$$\frac{\frac{}{\{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash(\text{fun } z\to!x)\Downarrow<\text{fun } z\to!x,\{x{:}\ell,y{:}\ell\}>,\{\ell{:}0\}}\qquad \mathbf{B}\qquad \mathbf{C}}{A = \{\ell{:}0\},\{x{:}\ell,y{:}\ell\}\vdash(\text{fun } z\to!x)(y:=!y+1)\Downarrow 1,\{\ell{:}1\}}$$

$$\frac{\frac{}{\{\ell{:}0\},\{x{:}\ell\}\vdash(\text{fun } y\to\ldots)\Downarrow<(\text{fun } y\to\ldots),\ \{x{:}\ell\}>,\{\ell{:}0\}}\qquad\frac{}{\{\ell{:}0\},\{x{:}\ell\}\vdash x\Downarrow\ell,\{\ell{:}0\}}\qquad \mathbf{A}}{\{\ell{:}0\},\{x{:}\ell\}\vdash(\text{fun } y\to(\text{fun } z\to!x)(y:=!y+1))x\Downarrow 1,\{\ell{:}1\}}$$

Example

$$\frac{\frac{}{\{\ell{:}1\},\{x{:}\ell,\ y{:}\ell,\ z{:}()\}\vdash x\Downarrow\ell,\{\ell{:}1\}}\qquad \{\ell{:}1\}(\ell)=1}{\mathbf{C} = \{\ell{:}1\},\{x{:}\ell,y{:}\ell,\ z{:}()\}\vdash !x\Downarrow 1,\{\ell{:}1\}}$$