

Operational Semantics of OCaml

- ▶ Present three systems
 - ▶ OS_{subst} : substitution model
 - ▶ OS_{clo} : closure model
 - ▶ OS_{state} : closure model, plus state
- ▶ In all systems, start with removing let's and letrec's as follows:
 - ▶ $\text{let } x = e \text{ in } e' \Rightarrow (\text{fun } x \rightarrow e')e$
 - ▶ $\text{let rec } f = e \text{ in } e' \Rightarrow (\text{fun } f \rightarrow e')(\text{rec } f = e)$
- ▶ Here, “ $\text{rec } f = e$ ” is a new expression added just for the operational semantics. Note: here, e must be an abstraction.

▶

OS_{subst}

- ▶ Just like OS_{simp} , but with recursion.
- ▶ Expressions:
 - ▶ constants (not higher order), var, application, abstraction, built-in function calls: $e_1 \oplus e_2$
 - ▶ Note: Partial application of built-ins is not allowed. This implies that in an application $e_1 e_2$, e_1 must reduce to a user-defined function.
- ▶ Values:
 - ▶ constants, (closed) abstractions
- ▶ Judgments:
 - ▶ $e \Downarrow v$ (e closed)

▶

Axioms

- ▶ (Const) $\overline{k \Downarrow k}$
- ▶ (Abstr) $\overline{\text{fun } x \rightarrow e \Downarrow \text{fun } x \rightarrow e}$
- ▶ (Rec) $\overline{\text{rec } f = e \Downarrow e [\text{rec } f = e / f]}$

Remember: e is an abstraction

Rules of inference (Same as OS_{simp})

- ▶ (δ)
$$\frac{e_1 \Downarrow v_1 \quad e_2 \Downarrow v_2 \quad v = v_1 \oplus v_2}{e_1 \oplus e_2 \Downarrow v}$$

- ▶ (if-true)
$$\frac{e_1 \Downarrow \text{true} \quad e_2 \Downarrow v}{\text{if } e_1 \text{ then } e_2 \text{ else } e_3 \Downarrow v}$$

- ▶ (if-false)
$$\frac{e_1 \Downarrow \text{false} \quad e_3 \Downarrow v}{\text{if } e_1 \text{ then } e_2 \text{ else } e_3 \Downarrow v}$$

Never gave these
in OS_{simp}

Continued

▶ (application)

$$\frac{e_1 \Downarrow \text{fun } x \rightarrow e \quad e_2 \Downarrow v \quad e[v/x] \Downarrow v'}{e_1 e_2 \Downarrow v'}$$



Example

let rec f = fun n -> if n=1 then 1 else n*f(n-1) in f(2) \Downarrow 2



OS_{clo}

- ▶ Same expressions; same translation of let and letrec.
- ▶ Definition:
 - ▶ Environments (notated η): map from variable to value
 - ▶ Closures = Expression \times Env (notated $\langle e, \eta \rangle$)

must be an
abstraction

must contain a value for every
free variable in expression

- ▶ Note that there is a circularity here: Env's contain closures and closures contain env's. We'll just keep this informal.



Axioms of OS_{clo}

- ▶ (Const) $\frac{}{\eta, k \Downarrow k}$ (Var) $\frac{}{\eta, x \Downarrow \eta(x)}$

- ▶ (Abstr) $\frac{}{\eta, \text{fun } x \rightarrow e \Downarrow \langle \text{fun } x \rightarrow e, \eta \rangle}$

- ▶ (Rec) $\frac{}{\eta, \text{rec } f = e \Downarrow \langle e, \eta' \rangle}$
where $\eta' = \eta[f \rightarrow \langle e, \eta' \rangle]$

Again, be informal about this.
Also, again note that e is an abstraction



Rules of inference

$$\triangleright (\delta) \quad \frac{\eta, e_1 \Downarrow v_1 \quad \eta, e_2 \Downarrow v_2 \quad v = v_1 \oplus v_2}{\eta, e_1 \oplus e_2 \Downarrow v}$$

\triangleright (App)

$$\frac{\eta, e_1 \Downarrow \langle \text{fun } x \rightarrow e, \eta' \rangle \quad \eta, e_2 \Downarrow v \quad \eta'[x \rightarrow v], e \Downarrow v'}{\eta, e_1 e_2 \Downarrow v'}$$

\triangleright

Example

$$\emptyset, (\text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3)4 \Downarrow 7$$

\triangleright

Example

$$\frac{[x:4], (\text{fun } y \rightarrow x+y) \Downarrow \langle \text{fun } y \rightarrow x+y, [x:4] \rangle \quad [x:4], 3 \Downarrow 3 \quad \mathbf{B}}{\mathbf{A} = [x:4], (\text{fun } y \rightarrow x+y)3 \Downarrow 7}$$

$$\frac{\emptyset, (\text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3) \Downarrow \langle \text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3, \emptyset \rangle \quad \emptyset, 4 \Downarrow 4 \quad \mathbf{A}}{\emptyset, (\text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3)4 \Downarrow 7}$$



Example

$$\frac{[x:4,y:3], x \Downarrow 4 \quad [x:4,y:3], y \Downarrow 3}{\mathbf{B} = [x:4,y:3], x+y \Downarrow 7}$$

$$\frac{[x:4], (\text{fun } y \rightarrow x+y) \Downarrow \langle \text{fun } y \rightarrow x+y, [x:4] \rangle \quad [x:4], 3 \Downarrow 3 \quad \mathbf{B}}{\mathbf{A} = [x:4], (\text{fun } y \rightarrow x+y)3 \Downarrow 7}$$

$$\frac{\emptyset, (\text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3) \Downarrow \langle \text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3, \emptyset \rangle \quad \emptyset, 4 \Downarrow 4 \quad \mathbf{A}}{\emptyset, (\text{fun } x \rightarrow (\text{fun } y \rightarrow x+y)3)4 \Downarrow 7}$$



OS_{state}

▶ Add:

- ▶ Locations
 - ▶ Notated l, l', l_1, etc
 - ▶ Infinite, unstructured set of atoms
- ▶ State
 - ▶ Notated σ
 - ▶ Map from locations to values

▶ Values: Constants, locations, closures

▶ Judgments:

$$\sigma, \eta \vdash e \Downarrow v, \sigma'$$



Axioms and rules of inference

- ▶ All rules are same as OS_{clo}, but “thread” state through subcomputation. States are never captured in closures.

▶ (Const) $\frac{}{\sigma, \eta \vdash k \Downarrow k, \sigma}$ (Var) $\frac{}{\sigma, \eta \vdash x \Downarrow \eta(x), \sigma}$

▶ (Abs) $\frac{}{\sigma, \eta \vdash \text{fun } x \rightarrow e \Downarrow \langle \text{fun } x \rightarrow e, \eta \rangle, \sigma}$

▶ (δ) $\frac{\sigma, \eta \vdash e_1 \Downarrow v_1, \sigma_1 \quad \sigma_1, \eta \vdash e_2 \Downarrow v_2, \sigma_2 \quad v = v_1 \oplus v_2}{\sigma, \eta \vdash e_1 \oplus e_2 \Downarrow v, \sigma_2}$



Axioms and rules of inference

- ▶ New rules for new operators

- ▶ (Deref)
$$\frac{\sigma, \eta \vdash e \Downarrow \ell, \sigma' \quad \ell \text{ a location} \quad \sigma'(\ell) = v}{\sigma, \eta \vdash !e \Downarrow v, \sigma'}$$

- ▶ (Assign)
$$\frac{\sigma, \eta \vdash e_1 \Downarrow \ell, \sigma' \quad \sigma', \eta \vdash e_2 \Downarrow v, \sigma''}{\sigma, \eta \vdash e_1 := e_2 \Downarrow (), \sigma''[\ell \rightarrow v]}$$

unique value of type unit

- ▶ (Ref)
$$\frac{\sigma, \eta \vdash e \Downarrow v, \sigma' \quad \ell \text{ a fresh location}}{\sigma, \eta \vdash \text{ref } e \Downarrow \ell, \sigma'[\ell \rightarrow v]}$$

▶

Example

$$\{\ell:0\}, \{x:\ell\} \vdash (\text{fun } y \rightarrow (\text{fun } z \rightarrow !x)(y := !y+1))x \Downarrow 1, \{\ell:1\}$$

▶

Example

$$\frac{\frac{}{\{l:0\}, \{x:l\} \vdash (\text{fun } y \rightarrow \dots) \Downarrow \langle (\text{fun } y \rightarrow \dots), \{x:l\} \rangle, \{l:0\}} \quad \frac{}{\{l:0\}, \{x:l\} \vdash x \Downarrow l, \{l:0\}} \quad \mathbf{A}}{\{l:0\}, \{x:l\} \vdash (\text{fun } y \rightarrow (\text{fun } z \rightarrow !x)(y := !y + 1))x \Downarrow 1, \{l:1\}}$$



Example

$$\frac{}{\{l:0\}, \{x:l, y:l\} \vdash (\text{fun } z \rightarrow !x) \Downarrow \langle \text{fun } z \rightarrow !x, \{x:l, y:l\} \rangle, \{l:0\}} \quad \mathbf{B} \quad \mathbf{C}$$

$$\mathbf{A} = \{l:0\}, \{x:l, y:l\} \vdash (\text{fun } z \rightarrow !x)(y := !y + 1) \Downarrow 1, \{l:1\}$$

$$\frac{\frac{}{\{l:0\}, \{x:l\} \vdash (\text{fun } y \rightarrow \dots) \Downarrow \langle (\text{fun } y \rightarrow \dots), \{x:l\} \rangle, \{l:0\}} \quad \frac{}{\{l:0\}, \{x:l\} \vdash x \Downarrow l, \{l:0\}} \quad \mathbf{A}}{\{l:0\}, \{x:l\} \vdash (\text{fun } y \rightarrow (\text{fun } z \rightarrow !x)(y := !y + 1))x \Downarrow 1, \{l:1\}}$$



Example

$$\mathbf{B} = \{\ell:0\}, \{x:\ell, y:\ell\} \vdash y := !y+1 \Downarrow 0, \{\ell:1\}$$

$$\frac{\{\ell:0\}, \{x:\ell, y:\ell\} \vdash (\text{fun } z \rightarrow !x) \Downarrow \langle \text{fun } z \rightarrow !x, \{x:\ell, y:\ell\} \rangle, \{\ell:0\}}{\mathbf{B} \quad \mathbf{C}}$$

$$\mathbf{A} = \{\ell:0\}, \{x:\ell, y:\ell\} \vdash (\text{fun } z \rightarrow !x)(y := !y+1) \Downarrow 1, \{\ell:1\}$$

$$\frac{\{\ell:0\}, \{x:\ell\} \vdash (\text{fun } y \rightarrow \dots) \Downarrow \langle (\text{fun } y \rightarrow \dots), \{x:\ell\} \rangle, \{\ell:0\} \quad \{\ell:0\}, \{x:\ell\} \vdash x \Downarrow \ell, \{\ell:0\}}{\mathbf{A}}$$

$$\{\ell:0\}, \{x:\ell\} \vdash (\text{fun } y \rightarrow (\text{fun } z \rightarrow !x)(y := !y+1))x \Downarrow 1, \{\ell:1\}$$



Example

$$\frac{\{\ell:0\}, \{x:\ell, y:\ell\} \vdash y \Downarrow \ell, \{\ell:0\} \quad \{\ell:0\}, \{x:\ell, y:\ell\} \vdash !y+1 \Downarrow 1, \{\ell:0\}}{\mathbf{B} = \{\ell:0\}, \{x:\ell, y:\ell\} \vdash y := !y+1 \Downarrow 0, \{\ell:1\}}$$

$$\frac{\{\ell:0\}, \{x:\ell, y:\ell\} \vdash (\text{fun } z \rightarrow !x) \Downarrow \langle \text{fun } z \rightarrow !x, \{x:\ell, y:\ell\} \rangle, \{\ell:0\}}{\mathbf{B} \quad \mathbf{C}}$$

$$\mathbf{A} = \{\ell:0\}, \{x:\ell, y:\ell\} \vdash (\text{fun } z \rightarrow !x)(y := !y+1) \Downarrow 1, \{\ell:1\}$$

$$\frac{\{\ell:0\}, \{x:\ell\} \vdash (\text{fun } y \rightarrow \dots) \Downarrow \langle (\text{fun } y \rightarrow \dots), \{x:\ell\} \rangle, \{\ell:0\} \quad \{\ell:0\}, \{x:\ell\} \vdash x \Downarrow \ell, \{\ell:0\}}{\mathbf{A}}$$

$$\{\ell:0\}, \{x:\ell\} \vdash (\text{fun } y \rightarrow (\text{fun } z \rightarrow !x)(y := !y+1))x \Downarrow 1, \{\ell:1\}$$



Example

$$\begin{array}{c}
 \frac{\frac{\frac{}{\{l:0\}, \{x:l, y:l\} \vdash y \Downarrow l, \{l:0\}} \quad \{l:0\}(l)=0}{\{l:0\}, \{x:l, y:l\} \vdash !y \Downarrow 0, \{l:0\}} \quad \frac{}{\{l:0\}, \{x:l, y:l\} \vdash 1 \Downarrow 1, \{l:0\}}}{\{l:0\}, \{x:l, y:l\} \vdash !y+1 \Downarrow 1, \{l:0\}} \\
 \hline
 \{l:0\}, \{x:l, y:l\} \vdash y \Downarrow l, \{l:0\} \quad \{l:0\}, \{x:l, y:l\} \vdash !y+1 \Downarrow 1, \{l:0\} \\
 \hline
 \mathbf{B} = \{l:0\}, \{x:l, y:l\} \vdash y := !y+1 \Downarrow 0, \{l:1\} \\
 \hline
 \{l:0\}, \{x:l, y:l\} \vdash (\text{fun } z \rightarrow !x) \Downarrow \langle \text{fun } z \rightarrow !x, \{x:l, y:l\} \rangle, \{l:0\} \quad \mathbf{B} \quad \mathbf{C} \\
 \hline
 \mathbf{A} = \{l:0\}, \{x:l, y:l\} \vdash (\text{fun } z \rightarrow !x)(y := !y+1) \Downarrow 1, \{l:1\} \\
 \hline
 \{l:0\}, \{x:l\} \vdash (\text{fun } y \rightarrow \dots) \Downarrow \langle (\text{fun } y \rightarrow \dots), \{x:l\} \rangle, \{l:0\} \quad \frac{}{\{l:0\}, \{x:l\} \vdash x \Downarrow l, \{l:0\}} \quad \mathbf{A} \\
 \hline
 \{l:0\}, \{x:l\} \vdash (\text{fun } y \rightarrow (\text{fun } z \rightarrow !x)(y := !y+1))x \Downarrow 1, \{l:1\} \\
 \hline
 \blacktriangleright
 \end{array}$$

Example

$$\begin{array}{c}
 \frac{\frac{}{\{l:1\}, \{x:l, y:l, z:()\} \vdash x \Downarrow l, \{l:1\}} \quad \{l:1\}(l)=1}{\mathbf{C} = \{l:1\}, \{x:l, y:l, z:()\} \vdash !x \Downarrow 1, \{l:1\}} \\
 \hline
 \blacktriangleright
 \end{array}$$