Programming Languages and Compilers (CS 421)

Elsa L Gunter
2112 SC, UIUC
http://courses.engr.illinois.edu/cs421

Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

Forward Recursion

- In Structural Recursion, split input into components and (eventually) recurse
- Forward Recursion form of Structural Recursion
- In forward recursion, first call the function recursively on all recursive components, and then build final result from partial results
- Wait until whole structure has been traversed to start building answer

Forward Recursion: Examples

```
# let rec double_up list =
    match list
    with [ ] -> [ ]
      | (x :: xs) -> (x :: x :: double_up xs);;
val double_up : 'a list -> 'a list = <fun>

# let rec poor_rev list =
   match list
   with [] -> []
     | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```


Forward Recursion: Examples

```
# let rec double_up list =
    match list
    with [ ] -> [ ]
      | (x :: xs) -> (x :: x :: double_up xs);;
val double_up : 'a list -> 'a list = <fun>
```

Base Case | Operator | Recursive Call

```
# let rec poor_rev list =
   match list
   with [] -> []
     | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```

Base Case | Operator | Recursive Call

Encoding Forward Recursion with Fold

```
# let rec append list1 list2 = match list1 with
  [ ] -> list2 | x::xs -> x :: append xs list2;;
val append : 'a list -> 'a list -> 'a list = <fun>
```

Base Case | Operation | Recursive Call

```
# let append list1 list2 =
   fold_right (fun x y -> x :: y) list1 list2;;
val append : 'a list -> 'a list -> 'a list = <fun>
# append [1;2;3] [4;5;6];;
 - : int list = [1; 2; 3; 4; 5; 6]
```


Mapping Recursion

- Can use the higher-order recursive map function instead of direct recursion

```
# let doubleList list =
   List.map (fun x -> 2 * x) list;;
val doubleList : int list -> int list = <fun>
# doubleList [2;3;4];;
- : int list = [4; 6; 8]
```

Continuations

- Idea: Use functions to represent the control flow of a program
- Method: Each procedure takes a function as an extra argument to which to pass its result; outer procedure "returns" no result
- Function receiving the result called a continuation
- Continuation acts as "accumulator" for work still to be done

Continuation Passing Style

- An expression is in **continuation passing style (CPS)** if every procedure call in it that is not directly a call to a continuation takes a continuation to which to give (pass) the result, and it returns no result (except the unknown ultimate result of the final continuation).

Recursive Functions

- Recall:

```
# let rec factorial n =
   if n = 0 then 1 else n * factorial (n - 1);;
 val factorial : int -> int = <fun>
# factorial 5;;
- : int = 120
```


Recursive Functions

```
# let rec factorial n =
   let b = (n = 0) in (* First computation *)
   if b then 1 (* Returned value *)
   else let s = n – 1 in (* Second computation *)
        let r = factorial s in  (* Third computation *)
        n * r (* Returned value *) ;;
val factorial : int -> int = <fun>
# factorial 5;;
- : int = 120
```


Recursive Functions

```
# let rec factorialk n k =
  eqk (n, 0)
  (fun b ->  (* First computation *)
   if b then k 1 (* Passed value *)
   else subk (n, 1)  (* Second computation *)
   (fun s -> factorialk s  (* Third computation *)
     (fun r -> timesk (n, r) k))) (* Passed value *)
val factorialk : int -> (int -> 'a) -> 'a = <fun>
# factorialk 5 report;;
120
- : unit = ()
```


Recursive Functions

- To make recursive call, must build intermediate continuation to
  - take recursive value: r
  - build it to final result: n * r
  - And pass it to final continuation:
  - times (n, r) k = k (n * r)

Example: CPS for length

```
let rec length list = match list with [] -> 0
   | (a :: bs) -> 1 + length bs
```

What is the let-expanded version of this?

Example: CPS for length

```
let rec length list = match list with [] -> 0
   | (a :: bs) -> 1 + length bs
```

What is the let-expanded version of this?

```
let rec length list = match list with [] -> 0
   | (a :: bs) -> let r1 = length bs in 1 + r1
```


Example: CPS for length

```
#let rec length list = match list with [] -> 0
   | (a :: bs) -> let r1 = length bs in 1 + r1
```

What is the CSP version of this?

Example: CPS for length

```
#let rec length list = match list with [] -> 0
   | (a :: bs) -> let r1 = length bs in 1 + r1
```

What is the CSP version of this?

```
#let rec lengthk list k = match list with [ ] -> k 0
  | x :: xs -> lengthk xs (fun r -> addk (r,1) k);;
val lengthk : 'a list -> (int -> 'b) -> 'b = <fun>
# lengthk [2;4;6;8] report;;
4
- : unit = ()
```


CPS for Higher Order Functions

- In CPS, every procedure / function takes a continuation to receive its result
- Procedures passed as arguments take continuations
- Procedures returned as results take continuations
- CPS version of higher-order functions must expect input procedures to take continuations

Example: all

```
#let rec all (p, l) = match l with [] -> true
   | (x :: xs) -> let b = p x in
       if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k =
```


Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->     true
```


Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
```


Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
 | (x :: xs) ->
```


Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
 | (x :: xs) -> pk x
```


Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
 | (x :: xs) -> pk x
           (fun b -> if b then                else
     )
```

Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
       if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
 | (x :: xs) -> pk x
         (fun b -> if b then allk (pk, xs) k else k
false)
val allk : ('a -> (bool -> 'b) -> 'b) * 'a list ->
(bool -> 'b) -> 'b = <fun>
```


CPS for sum

```
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
```


CPS for sum

```
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> let r1 = sum xs  in x + r1;;
```


CPS for sum

```
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> let r1 = sum xs  in x + r1;;
val sum : int list -> int = <fun>
# let rec sumk list k = match list with [ ] -> k 0
    | x :: xs -> sumk xs  (fun r1 -> addk x r1 k);;
```


CPS for sum

```
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> let r1 = sum xs  in x + r1;;
val sum : int list -> int = <fun>
# let rec sumk list k = match list with [ ] -> k 0
    | x :: xs -> sumk xs  (fun r1 -> addk (x, r1) k);;
val sumk : int list -> (int -> 'a) -> 'a = <fun>
# sumk [2;4;6;8] report;;
20
- : unit = ()
```


Terms

- A function is in Direct Style when it returns its result back to the caller.
- A Tail Call occurs when a function returns the result of another function call without any more computations (eg tail recursion)
- A function is in Continuation Passing Style when it, and every function call in it, passes its result to another function.
- Instead of returning the result to the caller, we pass it forward to another function.

Terminology

- Tail Position: A subexpression s of expressions e, such that if evaluated, will be taken as the value of e
  - if (x>3) then x + 2 else x - 4
  - let x = 5 in x + 4
- Tail Call: A function call that occurs in tail position
  - if (h x) then f x else (x + g x)

Terminology

- Available: A function call that can be executed by the current expression
- The fastest way to be unavailable is to be guarded by an abstraction (anonymous function, lambda lifted).
  - if (h x) then f x else (x + g x)
  - if (h x) then (fun x -> f x) else (g (x + x))

Not available

CPS Transformation

- Step 1: Add continuation argument to any function definition:
  - let f arg = e ⇒ let f arg k = e
  - Idea: Every function takes an extra parameter saying where the result goes
- Step 2: A simple expression in tail position should be passed to a continuation instead of returned:
  - return a ⇒ k a
  - Assuming a is a constant or variable.
  - "Simple" = "No available function calls."

CPS Transformation

- Step 3: Pass the current continuation to every function call in tail position
  - return f arg ⇒ f arg k
  - The function "isn't going to return," so we need to tell it where to put the result.

CPS Transformation

- Step 4: Each function call not in tail position needs to be converted to take a new continuation (containing the old continuation as appropriate)
  - return op (f arg) ⇒ f arg (fun r -> k(op r))
  - op represents a primitive operation
  - return f(g arg) ⇒ g arg (fun r-> f r k)

Example

Before:

```
let rec add_list lst =
match lst with
  [ ] -> 0
| 0 :: xs -> add_list xs
| x :: xs -> (+) x
   (add_list xs);;
```

After:

```
let rec add_listk lst k =
                    (* rule 1 *)
match lst with
| [ ] -> k 0 (* rule 2 *)
| 0 :: xs -> add_listk xs k
                    (* rule 3 *)
| x :: xs -> add_listk xs
         (fun r -> k ((+) x r));;
                 (* rule 4 *)
```

Other Uses for Continuations

- CPS designed to preserve order of evaluation
- Continuations used to express order of evaluation
- Can be used to change order of evaluation
- Implements:
  - Exceptions and exception handling
  - Co-routines
  - (pseudo, aka green) threads

Exceptions - Example

```
# exception Zero;;
exception Zero
# let rec list_mult_aux list =
   match list with [ ] -> 1
   | x :: xs ->
    if x = 0 then raise Zero
               else x * list_mult_aux xs;;
val list_mult_aux : int list -> int = <fun>
```


Exceptions - Example

```
# let list_mult list =
   try list_mult_aux list with Zero -> 0;;
val list_mult : int list -> int = <fun>
# list_mult [3;4;2];;
- : int = 24
# list_mult [7;4;0];;
- : int = 0
# list_mult_aux [7;4;0];;
Exception: Zero.
```


Exceptions

- When an exception is raised
  - The current computation is aborted
  - Control is “thrown” back up the call stack until a matching handler is found
  - All the intermediate calls waiting for a return values are thrown away

Implementing Exceptions

```
# let multkp (m, n) k =
   let r = m * n in
    (print_string "product result: ";
     print_int r; print_string "\n";
     k r);;
val multkp : int ( int -> (int -> 'a) -> 'a =
   <fun>
```


Implementing Exceptions

```
# let rec list_multk_aux list k kexcp =
   match list with [ ] -> k 1
   | x :: xs -> if x = 0 then  kexcp  0
     else list_multk_aux xs
          (fun r -> multkp (x, r) k) kexcp;;
val list_multk_aux : int list -> (int -> 'a) -> (int -> 'a)
  -> 'a = <fun>
# let rec list_multk list k = list_multk_aux list  k  k;;
val list_multk : int list -> (int -> 'a) -> 'a = <fun>
```

Implementing Exceptions

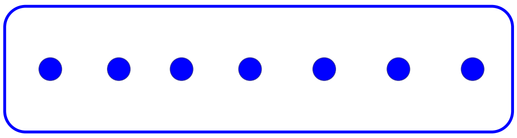
```
# list_multk [3;4;2] report;;
product result: 2
product result: 8
product result: 24
24
- : unit = ()
# list_multk [7;4;0] report;;
0
- : unit = ()
```


Variants - Syntax (slightly simplified)

- type *name* = C_1 [of ty_1] | . . . | C_n [of ty_n]
- Introduce a type called *name*
- (fun x -> C_i x) : ty_1 -> *name*
- C_i is called a *constructor*; if the optional type argument is omitted, it is called a *constant*
- Constructors are the basis of almost all pattern matching

Enumeration Types as Variants

An enumeration type is a collection of distinct values



In C and Ocaml they have an order structure; order by order of input

Enumeration Types as Variants

```
# type weekday = Monday | Tuesday | Wednesday
  | Thursday | Friday | Saturday | Sunday;;
type weekday =
    Monday
  | Tuesday
  | Wednesday
  | Thursday
  | Friday
  | Saturday
  | Sunday
```


Functions over Enumerations

```
# let day_after day = match day with
    Monday -> Tuesday
  | Tuesday -> Wednesday
  | Wednesday -> Thursday
  | Thursday -> Friday
  | Friday -> Saturday
  | Saturday -> Sunday
  | Sunday -> Monday;;
val day_after : weekday -> weekday = <fun>
```


Functions over Enumerations

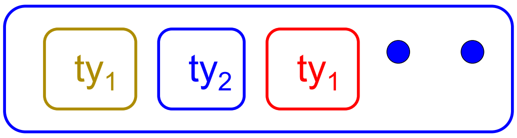
```
# let rec days_later n day =
    match n with 0 -> day
    | _ -> if n > 0
        then day_after (days_later (n - 1) day)
        else days_later (n + 7) day;;
val days_later : int -> weekday -> weekday
  = <fun>
```

Functions over Enumerations

```
# days_later 2 Tuesday;;
- : weekday = Thursday
# days_later (-1) Wednesday;;
- : weekday = Tuesday
# days_later (-4) Monday;;
- : weekday = Thursday
```


Disjoint Union Types

- Disjoint union of types, with some possibly occurring more than once



- We can also add in some new singleton elements

Disjoint Union Types

```
# type id = DriversLicense of int
  | SocialSecurity of int | Name of string;;
type id = DriversLicense of int | SocialSecurity
  of int | Name of string
# let check_id id = match id with
    DriversLicense num ->
     not (List.mem num [13570; 99999])
   | SocialSecurity num -> num < 900000000
   | Name str -> not (str = "John Doe");;
 val check_id : id -> bool = <fun>
```


Polymorphism in Variants

- The type 'a option is gives us something to represent non-existence or failure

```
# type 'a option = Some of 'a | None;;
type 'a option = Some of 'a | None
```

- Used to encode partial functions
- Often can replace the raising of an exception

Functions producing option

```
# let rec first p list =
   match list with [ ] -> None
   | (x::xs) -> if p x then Some x else first p xs;;
val first : ('a -> bool) -> 'a list -> 'a option = <fun>
# first (fun x -> x > 3) [1;3;4;2;5];;
- : int option = Some 4
# first (fun x -> x > 5) [1;3;4;2;5];;
- : int option = None
```


Functions over option

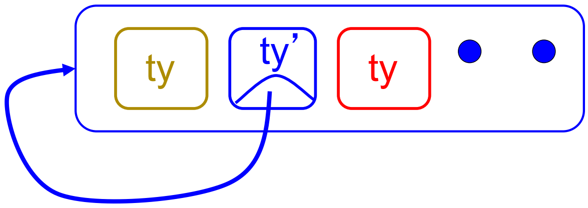
```
# let result_ok r =
   match r with None -> false
   | Some _ -> true;;
val result_ok : 'a option -> bool = <fun>
# result_ok (first (fun x -> x > 3) [1;3;4;2;5]);;
- : bool = true
# result_ok (first (fun x -> x > 5) [1;3;4;2;5]);;
- : bool = false
```

Folding over Variants

```
# let optionFold someFun noneVal opt =
    match opt with None -> noneVal
    | Some x -> someFun x;;
val optionFold : ('a -> 'b) -> 'b -> 'a option ->
  'b = <fun>
# let optionMap f opt =
    optionFold (fun x -> Some (f x)) None opt;;
val optionMap : ('a -> 'b) -> 'a option -> 'b
  option = <fun>
```


Recursive Types

- The type being defined may be a component of itself



Mapping over Variants

```
# let optionMap f opt =
    match opt with None -> None
    | Some x -> Some (f x);;
val optionMap : ('a -> 'b) -> 'a option -> 'b
  option = <fun>
# optionMap
  (fun x -> x - 2)
  (first (fun x -> x > 3) [1;3;4;2;5]);;
- : int option = Some 2
```

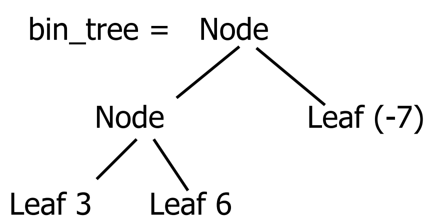

Recursive Data Types

```
# type int_Bin_Tree =
 Leaf of int | Node of (int_Bin_Tree *
  int_Bin_Tree);;

type int_Bin_Tree = Leaf of int | Node of
  (int_Bin_Tree * int_Bin_Tree)
```


Recursive Data Type Values

```
# let bin_tree =
 Node(Node(Leaf 3, Leaf 6),Leaf (-7));;

val bin_tree : int_Bin_Tree = Node (Node
  (Leaf 3, Leaf 6), Leaf (-7))
```


Recursive Data Type Values



Recursive Functions

```
# let rec first_leaf_value tree =
   match tree with (Leaf n) -> n
   | Node (left_tree, right_tree) ->
    first_leaf_value left_tree;;
val first_leaf_value : int_Bin_Tree -> int =
  <fun>
# let left = first_leaf_value bin_tree;;
val left : int = 3
```