Programming Languages and Compilers (CS 421)

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Based in part on slides by Mattox Beckman, as updated by Vikram Adve, Gul Agha, and Elsa L Gunter
Recall

```ocaml
# let rec poor_rev list = match list with
   | [] -> []
   | (x::xs) -> poor_rev xs @ [x];;

val poor_rev : 'a list -> 'a list = <fun>
```

What is its running time?
Comparison

- \( \text{poor_rev} [1,2,3] = \)
- \((\text{poor_rev} [2,3]) \@ [1] = \)
- \(((\text{poor_rev} [3]) \@ [2]) \@ [1] = \)
- \(((\text{poor_rev} [ ] ) \@ [3]) \@ [2]) \@ [1] = \)
- \((([ ] \@ [3]) \@ [2]) \@ [1]) = \)
- \((3:: ([ ] @ [2])) \@ [1] = \)
- \([3,2] \@ [1] = \)
- \(3 :: ([2] @ [1]) = \)
- \(3 :: (2:: ([ ] @ [1])) = [3, 2, 1] \)
Tail Recursion - Example

# let rec rev_aux list revlist = 
  match list with 
    [ ] -> revlist 
  | x :: xs -> rev_aux xs (x::revlist);;
val rev_aux : 'a list -> 'a list -> 'a list = <fun>

# let rev list = rev_aux list [ ];;
val rev : 'a list -> 'a list = <fun>

What is its running time?
Comparison

- \text{rev} [1,2,3] = 
- \text{rev\_aux} [1,2,3] [ ] = 
- \text{rev\_aux} [2,3] [1] = 
- \text{rev\_aux} [3] [2,1] = 
- \text{rev\_aux} [ ] [3,2,1] = [3,2,1]
Your turn now

Write a function

map_tail : (‘a -> ‘b) -> ‘a list -> ‘b list

that takes a function and a list of inputs and gives the result of applying the function on each argument, but in tail recursive form.

let map_tail f lst =
Folding - Tail Recursion

# let rec rev_aux list revlist =
  match list with
  [ ] -> revlist
  | x :: xs -> rev_aux xs (x::revlist);;
# let rev list = rev_aux list [ ];;;

# let rev list =
  fold_left
  (fun l -> fun x -> x :: l) (* comb op *)
  []
  list
  (* accumulator cell *)
Folding

- Can replace recursion by `fold_right` in any **forward primitive** recursive definition
  - Primitive recursive means it only recurses on immediate subcomponents of recursive data structure

- Can replace recursion by `fold_left` in any **tail primitive** recursive definition
Example of Tail Recursion

# let rec app fl x =
    match fl with [] -> x
    | (f :: rem_fs) -> f (app rem_fs x);;
val app : ('a -> 'a) list -> 'a -> 'a = <fun>

# let app fs x =
    let rec app_aux fl acc =
        match fl with [] -> acc
        | (f :: rem_fs) -> app_aux rem_fs
            (fun z -> acc (f z))
    in app_aux fs (fun y -> y) x;;
val app : ('a -> 'a) list -> 'a -> 'a = <fun>
Continuations

- A programming technique for all forms of “non-local” control flow:
  - non-local jumps
  - exceptions
  - general conversion of non-tail calls to tail calls
- Essentially it’s a higher-order function version of GOTO
Continuations

- **Idea:** Use functions to represent the control flow of a program

- **Method:** Each procedure takes a function as an extra argument to which to pass its result; outer procedure “returns” no result

- Function receiving the result called a **continuation**

- Continuation acts as “accumulator” for work still to be done
Simplest CPS Example

Identity function:
- `let ident x = x`

Identity function in CPS:
- `let identk x ret = ret x`
Example

- Simple function using a continuation:

```ocaml
# let addk (a, b) k = k (a + b);;

val addk : int * int -> (int -> 'a) -> 'a = <fun>
# addk (22, 20) report;;
42
- : unit = ()
```

- Simple reporting continuation:

```ocaml
# let report x = (print_int x; print_newline();
    exit 0);;

val report : int -> unit = <fun>
```
Continuation Passing Style

- Writing procedures such that all procedure calls take a continuation to which to give (pass) the result, and return no result, is called continuation passing style (CPS)
Continuation Passing Style

- A compilation technique to implement non-local control flow, especially useful in interpreters.

- A formalization of non-local control flow in denotational semantics

- Possible intermediate state in compiling functional code
Why CPS?

- Makes order of evaluation explicitly clear
- Allocates variables (to become registers) for each step of computation
- Essentially converts functional programs into imperative ones
  - Major step for compiling to assembly or byte code
- Tail recursion easily identified
- Strict forward recursion converted to tail recursion
  - At the expense of building large closures in heap
Other Uses for Continuations

- CPS designed to preserve order of evaluation
  - Continuations used to express order of evaluation

- Can be used to change order of evaluation

- Implements:
  - Exceptions and exception handling
  - Co-routines
  - (pseudo, aka green) threads
Example

- Simple reporting continuation:

```ocaml
# let report x = (print_int x; print_newline(); exit 0);;
val report : int -> unit = <fun>
```

- Simple function using a continuation:

```ocaml
# let addk (a, b) k = k (a + b);
val addk : int * int -> (int -> 'a) -> 'a = <fun>
# addk (22, 20) report;;
42
- : unit = ()
```
Simple Functions Taking Continuations

- Given a primitive operation, can convert it to pass its result forward to a continuation
- Examples:

```ocaml
# let subk (x, y) k = k (x - y);;
val subk : int * int -> (int -> 'a) -> 'a = <fun>
# let eqk (x, y) k = k (x = y);;
val eqk : 'a * 'a -> (bool -> 'b) -> 'b = <fun>
# let timesk (x, y) k = k (x * y);;
val timesk : int * int -> (int -> 'a) -> 'a = <fun>
```
Nesting Continuations

# let add_triple (x, y, z) = (x + y) + z;;
val add_triple : int * int * int -> int = <fun>

# let add_triple (x,y,z) = let p = x + y in p + z;;
val add_three : int -> int -> int -> int = <fun>

# let add_triple_k (x, y, z) k =
    addk (x, y) (fun p -> addk (p, z) k );;
val add_triple_k: int * int * int -> (int -> 'a) -> 'a = <fun>
add_three: a different order

# let add_triple_k (x, y, z) k =
   addk (x, y) (fun p -> addk (p, z) k );;

- How do we write add_triple_k to use a different order?
  # let add_triple (x, y, z) = x + (y + z);;

- let add_triple_k (x, y, z) k =
Terms

- A function is in **Direct Style** when it returns its result back to the caller.
- A **Tail Call** occurs when a function returns the result of another function call without any more computations (e.g., tail recursion).
- A function is in **Continuation Passing Style** when it, and every function call in it, passes its result to another function.
- Instead of returning the result to the caller, we pass it forward to another function.
Terminology

- Tail Position: A subexpression $s$ of expressions $e$, such that if evaluated, will be taken as the value of $e$
  - if $(x > 3)$ then $x + 2$ else $x - 4$
  - let $x = 5$ in $x + 4$

- Tail Call: A function call that occurs in tail position
  - if $(h \ x)$ then $f \ x$ else $(x + g \ x)$
Recall:

```ocaml
# let rec factorial n =
  if n = 0 then 1 else n * factorial (n - 1);;
val factorial : int -> int = <fun>

# factorial 5;;
- : int = 120
```
Recursive Functions

# let rec factorial n = 
   if n = 0 then 1 else n * factorial (n - 1);

# let rec factorial n = 
   let b = (n = 0) in (* 1st computation *) 
   if b then 1 (* Returned value *) 
   else let s = n - 1 in (* 2nd computation *) 
       let r = factorial s in (* 3rd computation *) 
       n * r (* Returned value *) ;;

val factorial : int -> int = <fun>

# factorial 5;;
- : int = 120
Recursive Functions

# let rec factorialk n k =
  eqk (n, 0)
  (fun b -> (* 1st computation *)
   if b then
     k 1 (* Passed val *)
   else
     subk (n,1) (* 2nd computation *)
     (fun s -> factorialk s (* 3rd computation*)
     (fun r -> timesk (n, r) k) (* Passed val*)
  )
)
val factorialk : int -> int = <fun>

# factorialk 5 report;;
120
Recursive Functions

- To make recursive call, must build intermediate continuation to
  - take recursive value: \( r \)
  - build it to final result: \( n \times r \)
  - And pass it to final continuation:
    - \( \text{times} \ (n, r) \ k = k \ (n \times r) \)
Example: CPS for length

let rec length list = match list with
  | [] -> 0
  | (a :: bs) -> 1 + length bs

What is the let-expanded version of this?
Example: CPS for length

```ocaml
let rec length list = match list with
  | [] -> 0
  | (a :: bs) -> 1 + length bs
```

What is the let-expanded version of this?

```ocaml
let rec length list = match list with
  | [] -> 0
  | (a :: bs) -> let r1 = length bs in
    1 + r1
```
Example: CPS for length

let rec length list = match list with
    [] -> 0
    | (a :: bs) -> 1 + length bs

What is the CPS version of this?
Example: CPS for length

```ocaml
let rec length list = match list with
  | [] -> 0
  | (a :: bs) -> 1 + length bs

What is the CPS version of this?

#let rec lengthk list k = match list with
  | [] -> k 0
  | x :: xs -> lengthk xs
      (fun r -> addk (r,1) k);;

# lengthk [2;4;6;8] report;;
4
```
Order of Evaluation Matters!

- Your turn (MP2): Write a function `quaddk` that takes three integer arguments, `a`, `b`, and `c`, and “returns” the result of the expression `\((2 \ast (a \ast b) + 5 \ast b) + c\)`
- `# let quaddk (a, b, c) k = ... ;;`

- Is the CPS form the same as for `\(2 \ast a \ast b + 5 \ast b + c\)`?
  - Refresher: Eval/App slides & Madhu’s notes posted on Piazza
  - **MP2 solutions implement the order of evaluation of arithmetic operators from right to left**
CPS for Higher Order Functions

- In CPS, every procedure / function takes a continuation to receive its result
- Procedures passed as arguments take continuations
- Procedures returned as results take continuations
- CPS version of higher-order functions must expect input procedures to take continuations
Example: all

```ocaml
#let rec all (p, l) = match l with [] -> true
 | (x :: xs) -> let b = p x in
    if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

What is the CPS version of this?
Example: all

```ocaml
let rec all (p, l) = match l with [] -> true
  | (x :: xs) -> let b = p x in
    if b then all (p, xs) else false
```

What is the CPS version of this?

```ocaml
let rec allk (pk, l) k =
```
Example: all

```ocaml
#let rec all (p, l) = match l with [] -> true
  | (x :: xs) -> let b = p x in
    if b then all (p, xs) else false
```

What is the CPS version of this?

```ocaml
#let rec allk (pk, l) k = match l with
  [] -> true
```

Example: all

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let rec all (p, l) = match l with [] -> true
  | (x :: xs) -> let b = p x in
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What is the CPS version of this?

```ocaml
let rec allk (pk, l) k = match l with [] -> k true
```

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Example: all

```ocaml
#let rec all (p, l) = match l with [] -> true
  | (x :: xs) -> let b = p x in
  if b then all (p, xs) else false
```

What is the CPS version of this?

```ocaml
#let rec allk (pk, l) k = match l with
  [] -> k true
  | (x :: xs) ->
```
Example: all

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      if b then all (p, xs) else false
```

What is the CPS version of this?

```ocaml
#let rec allk (pk, l) k = match l with
  [] -> k true
  | (x :: xs) -> pk x
```
Example: all

```ocaml
#let rec all (p, l) = match l with [] -> true
  | (x :: xs) -> let b = p x in
    if b then all (p, xs) else false
```

What is the CPS version of this?

```ocaml
#let rec allk (pk, l) k = match l with
  [] -> k true
  | (x :: xs) ->
    pk x (fun b -> if b then
             else
          )
```
Example: all

```ocaml
#let rec all (p, l) = match l with [] -> true
| (x :: xs) -> let b = p x in
  if b then all (p, xs) else false
```

- What is the CPS version of this?

```ocaml
#let rec allk (pk, l) k = match l with
  [] -> k true
| (x :: xs) -> pk x (fun b -> if b then allk (pk, xs) k else )
```
Example: all

```ocaml
#let rec all (p, l) = match l with [] -> true
  | (x :: xs) -> let b = p x in
    if b then all (p, xs) else false

What is the CPS version of this?
```

```ocaml
#let rec allk (pk, l) k = match l with
  [] -> k true
  | (x :: xs) ->
    pk x (fun b -> if b then allk (pk, xs) k
               else k false)

val allk : ('a -> (bool -> 'b) -> 'b) * 'a list ->
           (bool -> 'b) -> 'b = <fun>
```

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- Tail Position: A subexpression $s$ of expressions $e$, such that if evaluated, will be taken as the value of $e$
  - if $(x > 3)$ then $x + 2$ else $x - 4$
  - let $x = 5$ in $x + 4$

- Tail Call: A function call that occurs in tail position
  - if (h x) then $f x$ else $(x + g x)$
Terminology

- **Available**: A function call that can be executed by the current expression.

- The fastest way to be unavailable is to be guarded by an abstraction (anonymous function, lambda lifted).

  - if \((h \ x)\) then \(f \ x\) else \((x + g \ x)\)
  - if \((h \ x)\) then (fun \(x \rightarrow f \ x\)) else \((g \ (x + x))\)

Not available
CPS Transformation

- **Step 1**: Add continuation argument to any function definition:
  - `let f arg = e \Rightarrow let f arg k = e`
  - Idea: Every function takes an extra parameter saying where the result goes

- **Step 2**: A simple expression in tail position should be passed to a continuation instead of returned:
  - `return a \Rightarrow k a`
  - Assuming `a` is a constant or variable.
  - “Simple” = “No available function calls.”
CPS Transformation

- Step 3: Pass the current continuation to every function call in tail position
  - `return f arg \Rightarrow f arg k`
  - The function “isn’t going to return,” so we need to tell it where to put the result.
CPS Transformation

- Step 4: Each function call not in tail position needs to be converted to take a new continuation (containing the old continuation as appropriate)

  - return $op \ (f \ arg) \Rightarrow f \ arg \ (\text{fun } r -> k(op \ r))$

  - $op$ represents a primitive operation

  - return $f(g \ arg) \Rightarrow g \ arg \ (\text{fun } r -> f \ r \ k)$
Example

Step 1: Add continuation argument to any function definition
Step 2: A simple expression in tail position should be passed to a continuation instead of returned
Step 3: Pass the current continuation to every function call in tail position
Step 4: Each function call not in tail position needs to be converted to take a new continuation (containing the old continuation as appropriate)

Before:

```ocaml
let rec add_list lst =

match lst with
  | [] -> 0
| 0 :: xs -> add_list xs
| x :: xs -> (+) x
  (add_list xs);
```

After:

```ocaml
let rec add_listk lst k =

(* rule 1 *)
match lst with
  | [] -> k 0
| 0 :: xs -> add_listk xs k
  (* rule 2 *)
| x :: xs -> add_listk xs
  (fun r -> k ((+) x r));
  (* rule 3 *)
| x :: xs -> add_listk xs
  (fun r -> k ((+) x r));
  (* rule 4 *)
```
CPS for sum

# let rec sum list = match list with
    [ ] -> 0
  | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
CPS for sum

# let rec sum list = match list with
    [ ] -> 0
  | x :: xs -> x + sum xs ;;

# let rec sum list = match list with
    [ ] -> 0
  | x :: xs -> let r1 = sum xs in x + r1;;
CPS for sum

# let rec sum list = match list with
   [ ] -> 0
   | x :: xs -> x + sum xs ;;

# let rec sum list = match list with
   [ ] -> 0
   | x :: xs -> let r1 = sum xs in x + r1;;

# let rec sumk list k = match list with
   [ ] -> k 0
   | x :: xs -> sumk xs (fun r1 -> addk x r1 k);;
CPS for sum

# let rec sum list = match list with
  [ ] -> 0
  | x :: xs -> x + sum xs ;;

# let rec sum list = match list with
  [ ] -> 0
  | x :: xs -> let r1 = sum xs in x + r1;;

# let rec sumk list k = match list with
  [ ] -> k 0
  | x :: xs -> sumk xs (fun r1 -> addk x r1 k) ;;

# sumk [2;4;6;8] report ;;
20
Other Uses for Continuations

- CPS designed to **preserve evaluation order**
- **Continuations** used to **express** order of evaluation

- Can also be used to **change** order of evaluation

- Implements:
  - Exceptions and exception handling
  - Co-routines
  - (pseudo, aka green) threads
Exceptions - Example

# exception Zero;;
exception Zero

# let rec list_mult_aux list =
    match list with
    [ ] -> 1
    | x :: xs ->
        if x = 0 then raise Zero
        else x * list_mult_aux xs;;
val list_mult_aux : int list -> int = <fun>
Exceptions - Example

# let list_mult list =
   try list_mult_aux list with Zero -> 0;;
val list_mult : int list -> int = <fun>

# list_mult [3;4;2];;
- : int = 24

# list_mult [7;4;0];;
- : int = 0

# list_mult_aux [7;4;0];;
Exception: Zero.
Exceptions

- When an exception is raised
  - The current computation is aborted
  - Control is “thrown” back up the call stack until a matching handler is found
  - All the intermediate calls waiting for a return values are thrown away
Implementing Exceptions

```ml
# let multkp (m, n) k =
    let r = m * n in
    ( print_string "product result: ";
    print_int r; print_string "\n";
    k r);

val multkp : int ( int -> (int -> 'a) -> 'a = <fun>```
Implementing Exceptions

```ocaml
# let rec list_multk_aux list k kexcp =
    match list with
    [ ] -> k 1
    | x :: xs -> if x = 0 then kexcp 0 else
      list_multk_aux xs (fun r -> multkp (x, r) k)
    kexcp;;

# let rec list_multk list k =
    list_multk_aux list k (fun x -> print_string "nil\n");;
```
Implementing Exceptions

# list_multk [3;4;2] report;;
product result: 2
product result: 8
product result: 24
24
- : unit = ()

# list_multk [7;4;0] report;;
nil
- : unit = ()