Programming Languages and Compilers (CS 421)

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Based in part on slides by Mattox Beckman, as updated by Vikram Adve, Gul Agha, and Elsa L Gunter

Structural Recursion

- Functions on recursive datatypes (e.g., lists) tend to be recursive.
- Recursion over recursive datatypes generally by structural recursion:
  - Recursive calls made to components of structure of the same recursive type.
  - Base cases of recursive types stop the recursion of the function.
Structural Recursion : List Example

```plaintext
# let rec length list = match list with
  [ ] -> 0  (* Nil case *)
| x :: xs -> 1 + length xs;;  (* Cons case *)
val length : 'a list -> int = <fun>

# length [5; 4; 3; 2];;
- : int = 4
```

- Nil case [] is base case
- Cons case recurses on component list xs
Forward Recursion

- In Structural Recursion, split input into components and (eventually) recurse on components
- Forward Recursion form of Structural Recursion
- In forward recursion, first call the function recursively on all recursive components, and then build final result from partial results
- Wait until whole structure has been traversed to start building answer
Forward Recursion: Examples

```ocaml
# let rec double_up list = 
   match list with 
     [ ] -> [ ] 
     | (x :: xs) -> (x :: x :: double_up xs);;
val double_up : 'a list -> 'a list = <fun>

# let rec poor_rev list = 
   match list with 
     [] -> [] 
     | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```
Question

- How do you write length with forward recursion?

```ocaml
let rec length l =
```
Question

- How do you write length with forward recursion?

```ocaml
let rec length l =
    match l with [] ->
    | (a :: bs) ->
```
Question

- How do you write length with forward recursion?

```ml
let rec length l =
    match l with [] ->
    | (a :: bs) -> length bs
```
Question

- How do you write length with forward recursion?

```ml
let rec length l =
    match l with [] -> 0
    | (a :: bs) -> 1 + length bs
```
Functions Over Lists

# let rec double_up list =
  match list
  with [ ] -> [ ] (* pattern before ->, expression after *)
  | (x :: xs) -> (x :: x :: double_up xs);;
val double_up : 'a list -> 'a list = <fun>

# let fib5_2 = double_up fib5;;
val fib5_2 : int list = [8; 8; 5; 5; 3; 3; 2; 2; 1; 1; 1; 1]
Functions Over Lists

# let silly = double_up ["hi"; "there"];;

# let rec poor_rev list =
  match list with
    [] -> []
  | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>

# poor_rev silly;;
- : string list = ["there"; "there"; "hi"; "hi"]
An Important Optimization

- When a function call is made, the return address needs to be saved to the stack so we know to where to return when the call is finished.
- What if $f$ calls $g$ and $g$ calls $h$, but calling $h$ is the last thing $g$ does (a tail call)?
An Important Optimization

- When a function call is made, the return address needs to be saved to the stack so we know to where to return when the call is finished.
- What if \( f \) calls \( g \) and \( g \) calls \( h \), but calling \( h \) is the last thing \( g \) does (a *tail call*)?
- Then \( h \) can return directly to \( f \) instead of \( g \)
Tail Recursion

- A recursive program is tail recursive if all recursive calls are tail calls.
- Tail recursive programs may be optimized to be implemented as loops, thus removing the function call overhead for the recursive calls.
- Tail recursion generally requires extra “accumulator” arguments to pass partial results.
  - May require an auxiliary function.
Example of Tail Recursion

# let rec prod l =
  match l with [] -> 1
  | (x :: rem) -> x * prod rem;;
val prod : int list -> int = <fun>

# let prod list =
  let rec prod_aux l acc =
    match l with [] -> acc
    | (y :: rest) -> prod_aux rest (acc * y)
    (* Uses associativity of multiplication *)
    in prod_aux list 1;;
val prod : int list -> int = <fun>
Question

- How do you write length with tail recursion?

```haskell
let length l =
```
Question

- How do you write length with tail recursion?

```ocaml
let length l =
    let rec length_aux list n =
    in
```

9/14/2017
Question

- How do you write length with tail recursion?

```ml
let length l =
  let rec length_aux list n =
    match list with [] ->
    | (a :: bs) ->
  in
```
Question

How do you write length with tail recursion?

```ocaml
let length l =
    let rec length_aux list n =
        match list with [] -> n
        | (a :: bs) ->
    in
```

9/14/2017
Question

How do you write length with tail recursion?

```ml
let rec length_aux list n =
  match list with [] -> n
  | (a :: bs) -> length_aux
in
```

```ml
let length l =
  let rec length_aux list n =
    match list with [] -> n
    | (a :: bs) -> length_aux
  in
```
Question

- How do you write length with tail recursion?

```ocaml
let length l =
    let rec length_aux list n =
        match list with [] -> n
        | (a :: bs) -> length_aux bs
    in
```
Question

How do you write length with tail recursion?

```ocaml
let length l =
    let rec length_aux list n =
        match list with [] -> n
        | (a :: bs) -> length_aux bs (n + 1)
    in
```

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How do you write length with tail recursion?

```ml
let length l =
  let rec length_aux list n =
    match list with [] -> n
    | (a :: bs) -> length_aux bs (n + 1)
  in length_aux l 0
```
Mapping Recursion

- One common form of structural recursion applies a function to each element in the structure

```ocaml
# let rec doubleList list = match list with
  [ ] -> [ ]
| x::xs -> 2 * x :: doubleList xs;;
val doubleList : int list -> int list = <fun>

# doubleList [2;3;4];;
- : int list = [4; 6; 8]
```
Mapping Functions Over Lists

# let rec map f list =
    match list with
    [] -> []
  | (h::t) -> (f h) :: (map f t);
val map : ('a -> 'b) -> 'a list -> 'b list = <fun>

# map plus_two fib5;;
- : int list = [10; 7; 5; 4; 3; 3]

# map (fun x -> x - 1) fib6;;
: int list = [12; 7; 4; 2; 1; 0; 0]
Mapping Recursion

- Can use the higher-order recursive map function instead of direct recursion

```ocaml
# let doubleList list =  
    List.map (fun x -> 2 * x) list;;
val doubleList : int list -> int list = <fun>

# doubleList [2;3;4];;
- : int list = [4; 6; 8]
```

- Same function, but no rec
Your turn now

Write a function

make_app : (((‘a -> ‘b) * ‘a) list -> ‘b list

that takes a list of function – input pairs and gives the result of applying each function to its argument. Use map, no explicit recursion.

let make_app l =
Folding Recursion

Another common form “folds” an operation over the elements of the structure

```ocaml
# let rec multList list = match list with
  [ ] -> 1
| x::xs -> x * multList xs;;
val multList : int list -> int = <fun>

# multList [2;4;6];;
- : int = 48
```

Computes \((2 \times (4 \times (6 \times 1)))\)
Folding Functions over Lists

How are the following functions similar?

```ocaml
# let rec sumlist list = match list with
  | [] -> 0
  | x::xs -> x + sumlist xs;;

# sumlist [2;3;4];;
- : int = 9

# let rec prodlist list = match list with
  | [] -> 1
  | x::xs -> x * prodlist xs;;

# prodlist [2;3;4];;
- : int = 24
```
Folding Functions over Lists

How are the following functions similar?

```ocaml
# let rec sumlist list = match list with
    | [] -> 0
    | x::xs -> x + sumlist xs;;
# sumlist [2;3;4];;
- : int = 9

# let rec prodlist list = match list with
    | [] -> 1
    | x::xs -> x * prodlist xs;;
# prodlist [2;3;4];;
- : int = 24
```
How are the following functions similar?

```ocaml
# let rec sumlist list = match list with
  | [] -> 0
  | x::xs -> x + sumlist xs;;

# sumlist [2;3;4];;
- : int = 9

# let rec prodlist list = match list with
  | [] -> 1
  | x::xs -> x * prodlist xs;;

# prodlist [2;3;4];;
- : int = 24
```
How are the following functions similar?

```ocaml
# let rec sumlist list = match list with
  | [] -> 0
  | x::xs -> x + sumlist xs;;

# sumlist [2;3;4];;
- : int = 9

# let rec prodlist list = match list with
  | [] -> 1
  | x::xs -> x * prodlist xs;;

# prodlist [2;3;4];;
- : int = 24
```
Folding Functions over Lists

How are the following functions similar?

```ocaml
# let rec sumlist list = match list with
  | [] -> 0
  | x::xs -> x + sumlist xs;;

# sumlist [2;3;4];;
- : int = 9

# let rec prodlist list = match list with
  | [] -> 1
  | x::xs -> x * prodlist xs;;

# prodlist [2;3;4];;
- : int = 24
```

Combining Operator
Recursing over lists

# let rec fold_right f list b =
  match list with
    [] -> b
  | (x :: xs) -> f x (fold_right f xs b);

# fold_right
  (fun val init -> val + init)
  [1; 2; 3]
  0;;
- : int = 6
Recursing over lists

# let rec fold_right f list b =
  match list with
    [] -> b
  | (x :: xs) -> f x (fold_right f xs b);

# fold_right
  (fun s -> fun () -> print_string s)
    ["hi"; "there"]
  ();;
therehi- : unit = ()
Folding Recursion

- multList folds to the right
- Same as:

```ml
# let multList list =
    List.fold_right
        (fun x -> fun p -> x * p)
    list 1;;
val multList : int list -> int = <fun>

# multList [2;4;6];;
- : int = 48
```
Encoding Recursion with Fold

```ocaml
# let rec append list1 list2 = match list1 with
    [ ] -> list2 | x::xs -> x :: append xs list2;;
val append : 'a list -> 'a list -> 'a list = <fun>
```

Base Case  Operation  Recursive call

```ocaml
# let append list1 list2 =
    fold_right (fun x y -> x :: y) list1 list2;;
val append : 'a list -> 'a list -> 'a list = <fun>
# append [1;2;3] [4;5;6];;
- : int list = [1; 2; 3; 4; 5; 6]
```
Question

```ocaml
let rec length l =
  match l with 
  | []  -> 0
  | (a :: bs) -> 1 + length bs
```

- How do you write length with `fold_right`, but no explicit recursion?
let rec length l =
  match l with [] -> 0
  | (a :: bs) -> 1 + length bs

How do you write length with fold_right, but no explicit recursion?

let length list =
  List.fold_right (fun x -> fun n -> n + 1)
  list 0
Question

let rec length l =
    match l with [] -> 0
    | (a :: bs) -> 1 + length bs

- How do you write length with fold_right, but no explicit recursion?

let length list =
    List.fold_right (fun x -> fun n -> n + 1)
    list 0

Can you write fold_right (or fold_left) with just map? How, or why not?
Iterating over lists

```ocaml
# let rec fold_left f a list =
    match list with
    | [] -> a
    | (x :: xs) -> fold_left f (f a x) xs;;

val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a = <fun>

# fold_left
    (fun () -> print_string)
    ()
    ["hi"; "there"];;
hithere- : unit = ()
```
Encoding Tail Recursion with fold_left

```ocaml
# let prod list = let rec prod_aux l acc =
  match l with
  | [] -> acc
  | (y :: rest) -> prod_aux rest (acc * y)
  in prod_aux list 1;;

# let prod list =
  List.fold_left (fun acc y -> acc * y) 1 list;;
```

```
# prod [4;5;6];;
- : int = 120
```
Question

let length l =
  let rec length_aux list n =
  match list with [] -> n
  | (a :: bs) -> length_aux bs (n + 1)
in length_aux l 0

How do you write length with fold_left, but no explicit recursion?
Question

```ocaml
let length l =
  let rec length_aux list n =
    match list with [] -> n
    | (a :: bs) -> length_aux bs (n + 1)
  in length_aux l 0
```

How do you write length with fold_left, but no explicit recursion?

```ocaml
let length list =
  List.fold_left (fun n -> fun x -> n + 1)
  0 list
```
Folding

```ocaml
# let rec fold_left f a list = match list with
  [] -> a
| (x :: xs) -> fold_left f (f a x) xs;;

fold_left f a [x₁; x₂;…;xₙ] = f(...(f (f a x₁) x₂)...xₙ)
```

```ocaml
# let rec fold_right f list b = match list with
  [] -> b
| (x :: xs) -> f x (fold_right f xs b);;

fold_right f [x₁; x₂;…;xₙ] b = f x₁(f x₂ (...(f xₙ b)...))
```
Recall

```ocaml
# let rec poor_rev list = match list with
  | [] -> []
  | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```

What is its running time?
Quadratic Time

- Each step of the recursion takes time proportional to input
- Each step of the recursion makes only one recursive call.
- List example:

```ocaml
# let rec poor_rev list = match list with
  | [] -> []
  | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```
Comparison

- \text{poor\_rev} [1,2,3] =
- (poor\_rev [2,3]) @ [1] =
- (((poor\_rev [3]) @ [2]) @ [1] =
- (((poor\_rev [ ])) @ [3]) @ [2]) @ [1] =
- ([[ ] @ [3]) @ [2]) @ [1]) =
- ([3] @ [2]) @ [1] =
- (3:: ([ ] @ [2])) @ [1] =
- [3,2] @ [1] =
- 3 :: ([2] @ [1]) =
- 3 :: (2:: ([ ] @ [1]))) = [3, 2, 1]
Tail Recursion - Example

```ocaml
# let rec rev_aux list revlist =
  match list with
  [ ] -> revlist
| x :: xs -> rev_aux xs (x::revlist);
val rev_aux : 'a list -> 'a list -> 'a list = <fun>

# let rev list = rev_aux list [ ];
val rev : 'a list -> 'a list = <fun>
```

- What is its running time?
Comparison

- $\text{rev} [1,2,3] = \quad$  
- $\text{rev}_\text{aux} [1,2,3] [ ] = \quad$  
- $\text{rev}_\text{aux} [2,3] [1] = \quad$  
- $\text{rev}_\text{aux} [3] [2,1] = \quad$  
- $\text{rev}_\text{aux} [ ] [3,2,1] = [3,2,1]$
Folding - Tail Recursion

# let rec rev_aux list revlist =
  match list with
  | [] -> revlist
  | x :: xs -> rev_aux xs (x::revlist);

# let rev list = rev_aux list [];;

# let rev list =
  fold_left
  (fun l -> fun x -> x :: l) (* comb op *)
  [] (* accumulator cell *)
  list
Folding

- Can replace recursion by `fold_right` in any **forward primitive** recursive definition
  - Primitive recursive means it only recurses on immediate subcomponents of recursive data structure
- Can replace recursion by `fold_left` in any **tail primitive** recursive definition
Example of Tail Recursion

```ocaml
# let rec app fl x =
  match fl with [ ] -> x
  | (f :: rem_fs) -> f (app rem_fs x);;
val app : ('a -> 'a) list -> 'a -> 'a = <fun>

# let app fs x =
  let rec app_aux fl acc =
    match fl with [ ] -> acc
    | (f :: rem_fs) -> app_aux rem_fs
      (fun z -> acc (f z))
  in app_aux fs (fun y -> y) x;;
val app : ('a -> 'a) list -> 'a -> 'a = <fun>
```
Continuation Passing Style

- A programming technique for all forms of “non-local” control flow:
  - non-local jumps
  - exceptions
  - general conversion of non-tail calls to tail calls
- Essentially it’s a higher-order function version of GOTO
Continuations

- Idea: Use functions to represent the control flow of a program
- Method: Each procedure takes a function as an argument to which to pass its result; outer procedure “returns” no result
- Function receiving the result called a continuation
- Continuation acts as “accumulator” for work still to be done
Continuation Passing Style

- Writing procedures so that they take a continuation to which to give (pass) the result, and return no result, is called continuation passing style (CPS)
Example

- Simple reporting continuation:

```ocaml
# let report x = (print_int x;
    print_newline( ) );;;
val report : int -> unit = <fun>
```

- Simple function using a continuation:

```ocaml
# let plusk a b k = k (a + b)
val plusk : int -> int -> (int -> 'a) -> 'a = <fun>
# plusk 20 22 report;;
42
- : unit = ()
```
Example of Tail Recursion & CSP

```ocaml
# let app fs x = 
  let rec app_aux fl acc = 
    match fl with 
      [] -> acc 
    | (f :: rem_fs) -> app_aux rem_fs (fun z -> acc (f z)) 
  in app_aux fs (fun y -> y) x;;
val app : ('a -> 'a) list -> 'a -> 'a = <fun>

# let rec appk fl x k = 
  match fl with 
    [] -> k x 
  | (f :: rem_fs) -> appk rem_fs x (fun z -> k (f z));;
val appk : ('a -> 'a) list -> 'a -> ('a -> 'b) -> 'b
```
Example of Tail Recursion & CSP

# let rec appk fl x k =
   match fl with
     []  -> k x
   | (f :: rem_fs)  -> appk rem_fs x (fun z -> k (f z));;

# appk [(fun x->x+1); (fun x -> x*5)] 2 (fun x->x);;
- : int = 11
Continuation Passing Style

- A compilation technique to implement non-local control flow, especially useful in interpreters.

- A formalization of non-local control flow in denotational semantics

- Possible intermediate state in compiling functional code
Terms

- A function is in **Direct Style** when it returns its result back to the caller.

- A **Tail Call** occurs when a function returns the result of another function call without any more computations (e.g. tail recursion)

- A function is in **Continuation Passing Style** when it passes its result to another function.
  - Instead of returning the result to the caller, we pass it forward to another function.
Continuation Passing Style

- A compilation technique to implement non-local control flow, especially useful in interpreters.

- A formalization of non-local control flow in denotational semantics

- Possible intermediate state in compiling functional code
Example

- Simple reporting continuation:

```ocaml
# let report x = (print_int x;
    print_newline( ) );;;
val report : int -> unit = <fun>
```

- Simple function using a continuation:

```ocaml
# let plusk a b k = k (a + b)
val plusk : int -> int -> (int -> 'a) -> 'a
    = <fun>
# plusk 20 22 report;;
42
- : unit = ()
```
Simple Functions Taking Continuations

- Given a primitive operation, can convert it to pass its result forward to a continuation

- Examples:

```ocaml
# let subk x y k = k(x + y);;
val subk : int -> int -> (int -> 'a) -> 'a = <fun>

# let eqk x y k = k(x = y);;
val eqk : 'a -> 'a -> (bool -> 'b) -> 'b = <fun>

# let timesk x y k = k(x * y);;
val timesk : int -> int -> (int -> 'a) -> 'a = <fun>
```
Nesting Continuations

# let add_three x y z = x + y + z;;
val add_three : int -> int -> int -> int = <fun>

# let add_three x y z = let p = x + y in p + z;;
val add_three : int -> int -> int -> int = <fun>

# let add_three_k x y z k =
    addk x y (fun p -> addk p z k);;
val add_three_k : int -> int -> int -> (int -> 'a -> 'a) -> 'a = <fun>