Nested Recursive Types

```
# type 'a labeled_tree = 
  TreeNode of ('a * 'a labeled_tree list);;

type 'a labeled_tree = TreeNode of ('a * 'a labeled_tree list)
```

Nested Recursive Type Values

```
# let ltree =
  TreeNode(5,
    [TreeNode (3, []);
     TreeNode (2, [TreeNode (1, []);
                   TreeNode (7, [])]);
     TreeNode (5, [])]);;
```

Nested Recursive Type Values

```
val ltree : int labeled_tree = 
  TreeNode 
   (5, 
    [TreeNode (3, []); TreeNode (2, 
      [TreeNode (1, []); TreeNode (7, [])]); 
     TreeNode (5, [])])
```

Nested Recursive Type Values

```
Ltree =  TreeNode(5) 
          ::                ::                 ::       ... 
         
                TreeNode(1)  TreeNode(7) 
         
                       [ ]              [ ]
```

Nested Recursive Type Values

```
5

3           2           5  

1           7
```
Mutually Recursive Functions

```ocaml
# let rec flatten_tree labtree = match labtree with TreeNode (x,treelist) -> x::flatten_tree_list treelist and flatten_tree_list treelist = match treelist with [] -> [] | labtree::labtrees -> flatten_tree labtree @ flatten_tree_list labtrees;;
```

val flatten_tree : 'a labeled_tree -> 'a list = <fun>
val flatten_tree_list : 'a labeled_tree list -> 'a list = <fun>

```
# flatten_tree ltree;;
- : int list = [5; 3; 2; 1; 7; 5]
```

Nested recursive types lead to mutually recursive functions

Why Data Types?

- Data types play a key role in:
  - Data abstraction in the design of programs
  - Type checking in the analysis of programs
  - Compile-time code generation in the translation and execution of programs
  - Data layout (how many words; which are data and which are pointers) dictated by type

Terminology

- Type: A type \( t \) defines a set of possible data values
- E.g. short in C is \( \{x | 2^{15} - 1 \geq x \geq -2^{15}\} \)
- A value in this set is said to have type \( t \)
- Type system: rules of a language assigning types to expressions

Types as Specifications

- Types describe properties
- Different type systems describe different properties, eg
  - Data is read-write versus read-only
  - Operation has authority to access data
  - Data came from “right” source
  - Operation might or could not raise an exception
- Common type systems focus on types describing same data layout and access methods

Sound Type System

- If an expression is assigned type \( t \), and it evaluates to a value \( v \), then \( v \) is in the set of values defined by \( t \)
- SML, OCAML, Scheme and Ada have sound type systems
- Most implementations of C and C++ do not
Strongly Typed Language

- When no application of an operator to arguments can lead to a run-time type error, language is strongly typed
  - Eg: 1 + 2.3;;
- Depends on definition of “type error”

C++ claimed to be “strongly typed”, but
- Union types allow creating a value at one type and using it at another
- Type coercions may cause unexpected (undesirable) effects
- No array bounds check (in fact, no runtime checks at all)

SML, OCAML “strongly typed” but still must do dynamic array bounds checks, runtime type case analysis, and other checks

Static vs Dynamic Types

- Static type: type assigned to an expression at compile time
- Dynamic type: type assigned to a storage location at run time
- Statically typed language: static type assigned to every expression at compile time
- Dynamically typed language: type of an expression determined at run time

Type Checking

- When is op(arg1,…,argn) allowed?
- Type checking assures that operations are applied to the right number of arguments of the right types
  - Right type may mean same type as was specified, or may mean that there is a predefined implicit coercion that will be applied
- Used to resolve overloaded operations

Type Checking

- Type checking may be done statically at compile time or dynamically at run time
- Dynamically typed (aka untyped) languages (eg LISP, Prolog) do only dynamic type checking
- Statically typed languages can do most type checking statically

Dynamic Type Checking

- Performed at run-time before each operation is applied
- Types of variables and operations left unspecified until run-time
  - Same variable may be used at different types
Dynamic Type Checking

- Data object must contain type information
- Errors aren’t detected until violating application is executed (maybe years after the code was written)

Static Type Checking

- Performed after parsing, before code generation
- Type of every variable and signature of every operator must be known at compile time

Static Type Checking

- Can eliminate need to store type information in data object if no dynamic type checking is needed
- Catches many programming errors at earliest point
- Can’t check types that depend on dynamically computed values
  - Eg: array bounds

Static Type Checking

- Typically places restrictions on languages
  - Garbage collection
  - References instead of pointers
  - All variables initialized when created
  - Variable only used at one type
  - Union types allow for work-arounds, but effectively introduce dynamic type checks

Type Declarations

- Type declarations: explicit assignment of types to variables (signatures to functions) in the code of a program
  - Must be checked in a strongly typed language
  - Often not necessary for strong typing or even static typing (depends on the type system)

Type Inference

- Type inference: A program analysis to assign a type to an expression from the program context of the expression
  - Fully static type inference first introduced by Robin Miller in ML
  - Haskle, OCAML, SML all use type inference
  - Records are a problem for type inference
**Format of Type Judgments**

- A **type judgement** has the form \( \Gamma |- \text{exp} : \tau \)
- \( \Gamma \) is a typing environment
- Supplies the types of variables and functions
- \( \Gamma \) is a set of the form \( \{ x : \sigma, \ldots \} \)
  - For any \( x \) at most one \( \sigma \) such that \( (x : \sigma) \in \Gamma \)
- \text{exp} is a program expression
- \( \tau \) is a type to be assigned to \text{exp}
- \( |- \) pronounced “turnstyle”, or “entails” (or “satisfies” or, informally, “shows”)

**Axioms - Constants**

\[ \Gamma |- n : \text{int} \quad (\text{assuming } n \text{ is an integer constant}) \]

\[ \Gamma |- \text{true} : \text{bool} \quad \Gamma |- \text{false} : \text{bool} \]

- These rules are true with any typing environment
- \( \Gamma, n \) are meta-variables

**Axioms – Variables (Monomorphic Rule)**

Notation: Let \( \Gamma(x) = \sigma \) if \( x : \sigma \in \Gamma \)

Note: if such \( \sigma \) exits, its unique

Variable axiom:

\[ \Gamma |- x : \sigma \quad \text{if } \Gamma(x) = \sigma \]

**Simple Rules - Arithmetic**

Primitive operators ( \( \oplus \in \{ +, -, *, \ldots \} \)):

\[ \Gamma |- e_1 : \tau_1 \quad \Gamma |- e_2 : \tau_2 \quad (\oplus) : \tau_1 \rightarrow \tau_2 \rightarrow \tau_3 \]

\[ \Gamma |- e_1 \oplus e_2 : \tau_3 \]

Relations ( \( \sim \in \{ <, >, =, <=, >= \} \)):

\[ \Gamma |- e_1 : \tau \quad \Gamma |- e_2 : \tau \]

\[ \Gamma |- e_1 \sim e_2 : \text{bool} \]

For the moment, think \( \tau \) is \text{int}

**Example:** \{\( x \text{ : int} \)\} |- \( x + 2 = 3 \) : \text{bool}

What do we need for the left side?

\{\( x \text{ : int} \)\} |- \( x + 2 = 3 \) : \text{bool}

What do we need for the left side?

\{\( x \text{ : int} \)\} |- \( x + 2 \) : \text{int} \quad \{\( x \text{ : int} \)\} |- \( 3 \) : \text{int}

\{\( x \text{ : int} \)\} |- \( x + 2 = 3 \) : \text{bool}
Example: \{x:int\} |- x + 2 = 3 : bool

How to finish?

{\{x : int\} |- x + 2 : int \valid \{x:int\} |- 3 : int} 

Complete Proof (type derivation)

\[\begin{align*} \{x : int\} &| - x + 2 : int \valid \{x : int\} &| - 3 : int \valid \{x : int\} &| - x + 2 = 3 : bool \end{align*} \]

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Simple Rules - Booleans

Connectives

\[\begin{align*} \varGamma &| - e_1 : \text{bool} \valid \varGamma &| - e_2 : \text{bool} \valid \varGamma &| - e_1 \&\& e_2 : \text{bool} \end{align*} \]

\[\begin{align*} \varGamma &| - e_1 : \text{bool} \valid \varGamma &| - e_2 : \text{bool} \valid \varGamma &| - e_1 || e_2 : \text{bool} \end{align*} \]

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Function Application

- Application rule:

\[\begin{align*} \varGamma &| - e_1 : \tau_1 \to \tau_2 \valid \varGamma &| - e_2 : \tau_1 \valid \varGamma &| - (e_1 \, e_2) : \tau_2 \end{align*} \]

- If you have a function expression \(e_1\) of type \(\tau_1 \to \tau_2\) applied to an argument \(e_2\) of type \(\tau_1\), the resulting expression \(e_1 \, e_2\) has type \(\tau_2\)

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Fun Rule

- Rules describe types, but also how the environment \(\varGamma\) may change

- Can only do what rule allows!

- fun rule:

\[\begin{align*} \{x : \tau_1\} \} + \varGamma &| - e : \tau_2 \valid \varGamma &| - \text{fun } x \to e : \tau_1 \to \tau_2 \end{align*} \]
Fun Examples

\{y : \text{int}\} + \Gamma |- y + 3 : \text{int} \\
\Gamma |- \text{fun } y \rightarrow y + 3 : \text{int} \rightarrow \text{int}

\{f : \text{int} \rightarrow \text{bool}\} + \Gamma |- f 2 :: [\text{true}] : \text{bool list} \\
\Gamma |- (\text{fun } f \rightarrow f 2 :: [\text{true}]) : (\text{int} \rightarrow \text{bool}) \rightarrow \text{bool list}

(Monomorphic) Let and Let Rec

- let rule:
  \[
  \Gamma |- e_1 : \tau_1 \quad \{x : \tau_1\} + \Gamma |- e_2 : \tau_2 \\
  \Gamma |- (\text{let } x = e_1 \text{ in } e_2) : \tau_2
  \]

- let rec rule:
  \[
  \{x : \tau_1\} + \Gamma |- e_1 : \tau_1 \quad \{x : \tau_1\} + \Gamma |- e_2 : \tau_2 \\
  \Gamma |- (\text{let rec } x = e_1 \text{ in } e_2) : \tau_2
  \]

Example

- Which rule do we apply?

\[
\Gamma |- (\text{let rec one = 1 :: one in} \\
\text{let x = 2 in} \\
\text{fun y -> (x :: y :: one)}) : \text{int} \rightarrow \text{int list}
\]

Proof of 1

- Which rule?

\[
\{\text{one : int list}\} |- (1 :: \text{one}) : \text{int list}
\]
Proof of 3

Constants Rule
\[ \{\text{one : int list}\} \vdash \]
\[ \text{ (::) : int \rightarrow int list \rightarrow int list} \]
\[ \{\text{one : int list}\} \vdash \text{ (::) 1 : int list \rightarrow int list} \]

Proof of 4

Rule for variables
\[ \{\text{one : int list}\} \vdash \text{ one : int list} \]

Proof of 2

5 \quad \{\text{x:int; one : int list}\} \vdash 

\[ \text{ fun y \rightarrow} \]
\[ \text{ (x :: y :: one))} \]
\[ \{\text{one : int list}\} \vdash \text{ 2:int \rightarrow int list} \]
\[ \{\text{one : int list}\} \vdash \text{ (let x = 2 in} \]
\[ \text{ fun y \rightarrow (x :: y :: one)) : int \rightarrow int list} \]

Proof of 5

6 \quad \{\text{y:int; x:int; one : int list}\} \vdash 

\[ \text{ (::) : int \rightarrow int list \rightarrow int list} \]
\[ \{\text{y:int; x:int; one : int list}\} \vdash \text{ (::) 1 : int list \rightarrow int list} \]
\[ \{\text{y:int; x:int; one : int list}\} \vdash \text{ fun y \rightarrow (x :: y :: one)) : int \rightarrow int list} \]

7 \quad \{\text{y:int; x:int; one : int list}\} \vdash 

\[ \text{ (::) : int \rightarrow int list \rightarrow int list} \]
\[ \{\text{y:int; x:int; one : int list}\} \vdash \text{ (::) 1 : int list \rightarrow int list} \]
\[ \{\text{y:int; x:int; one : int list}\} \vdash \text{ fun y \rightarrow (x :: y :: one)) : int \rightarrow int list} \]
**Proof of 6**

<table>
<thead>
<tr>
<th>Constant</th>
<th>Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>{...}</td>
<td>- (::)</td>
</tr>
<tr>
<td>: int→ int list→ int list</td>
<td>{...; x:int;...}</td>
</tr>
<tr>
<td>{y:int; x:int; one : int list}</td>
<td>- ( (::) x)</td>
</tr>
<tr>
<td>:int list→ int list</td>
<td></td>
</tr>
</tbody>
</table>

**Proof of 7**

Pf of 6 [y/x] | Variable |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>{y:int; ...}</td>
<td>- ( (::) y)</td>
</tr>
<tr>
<td>{…; one: int list}</td>
<td>-</td>
</tr>
<tr>
<td>:int list→ int list</td>
<td></td>
</tr>
<tr>
<td>one: int list</td>
<td></td>
</tr>
<tr>
<td>{y:int; x:int; one : int list}</td>
<td>- (y :: one) : int list</td>
</tr>
<tr>
<td>:int list→ int list</td>
<td></td>
</tr>
</tbody>
</table>

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**Curry - Howard Isomorphism**

- Type Systems are logics; logics are type systems
- Types are propositions; propositions are types
- Terms are proofs; proofs are terms
- Function space arrow corresponds to implication; application corresponds to modus ponens

**Curry - Howard Isomorphism**

- Modus Ponens

\[
\begin{align*}
A \Rightarrow B & \quad A \\
\hline
B & \\
\end{align*}
\]

- Application

\[
\begin{align*}
\Gamma |\!\!\!| e_1 : \alpha \Rightarrow \beta & \quad \Gamma |\!\!\!| e_2 : \alpha \\
\hline
\Gamma |\!\!\!| (e_1 \ e_2) : \beta \\
\end{align*}
\]

**Mia Copa**

- The above system can’t handle polymorphism as in OCAML
- No type variables in type language (only meta-variable in the logic)
- Would need:
  - Object level type variables and some kind of type quantification
  - let and let rec rules to introduce polymorphism
  - Explicit rule to eliminate (instantiate) polymorphism