More on SAT

Lecture 25

April 27, 2023

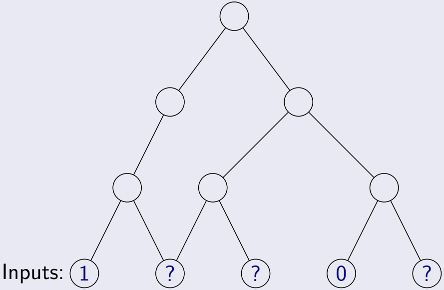
Part I

Circuit SAT

Circuits

Definition

A circuit is a directed *acyclic* graph with



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2. Every other vertex is labelled $\vee$, $\wedge$ or $\neg$.
3. Single node output vertex with no outgoing edges.

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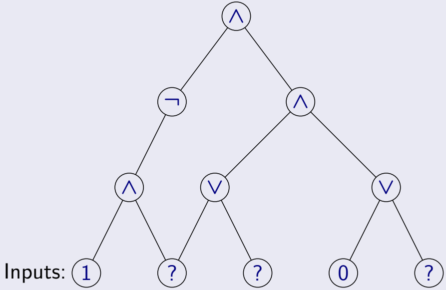

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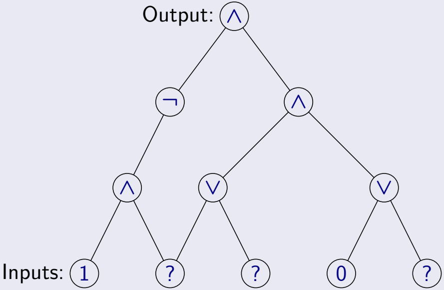

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CSAT: Circuit Satisfaction

Definition (Circuit Satisfaction (CSAT).)

Given a circuit as input, is there an assignment to the input variables that causes the output to get value 1?

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Claim

CSAT *is in* **NP**.

1. Certificate: Assignment to input variables.
2. Certifier: Evaluate the value of each gate in a topological sort of DAG and check the output gate value.

Circuit SAT vs SAT

CNF formulas are a rather restricted form of Boolean formulas.

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CNF formulas are a rather restricted form of Boolean formulas.

Circuits are a much more powerful (and hence easier) way to express Boolean formulas

However they are equivalent in terms of polynomial-time solvability.

Theorem

SAT $\leq_P$ **3SAT** $\leq_P$ **CSAT**.

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CSAT $\leq_P$ **SAT** $\leq_P$ **3SAT**.

Converting a CNF formula into a Circuit

Given $3\mathrm{CNF}$ formulat φ with n variables and m clauses, create a Circuit C.

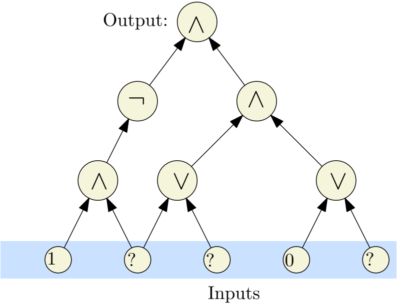
- Inputs to C are the n boolean variables $x_1, x_2, \ldots, x_n$
- Use NOT gate to generate literal $\neg x_i$ for each variable x_i
- For each clause $(\ell_1 \vee \ell_2 \vee \ell_3)$ use two OR gates to mimic formula
- Combine the outputs for the clauses using AND gates to obtain the final output

Example

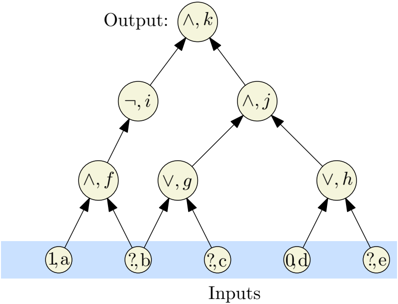
$$\varphi = \left(x_1 \vee \vee x_3 \vee x_4\right) \wedge \left(x_1 \vee \neg x_2 \vee \neg x_3\right) \wedge \left(\neg x_2 \vee \neg x_3 \vee x_4\right)$$

Converting a circuit into a CNF formula

Label the nodes



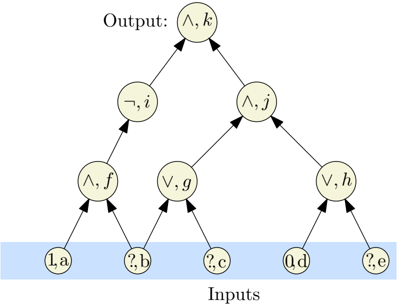
(A) Input circuit



(B) Label the nodes.

Converting a circuit into a CNF formula

Introduce a variable for each node



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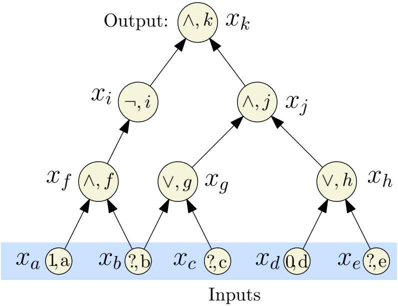
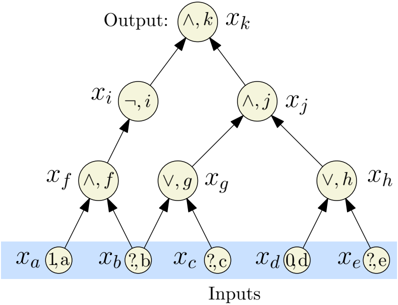


(C) Introduce var for each node.

Converting a circuit into a CNF formula

Write a sub-formula for each variable that is true if the var is computed correctly.



(C) Introduce var for each node.

x_k (Demand a sat' assignment!)

$x_k = x_i \wedge x_j$

$x_j = x_g \wedge x_h$

$x_i = \neg x_f$

$x_h = x_d \vee x_e$

$x_g = x_b \vee x_c$

$x_f = x_a \wedge x_b$

$x_d = 0$

$x_a = 1$

(D) Write a sub-formula for each variable that is true if the var is computed correctly.

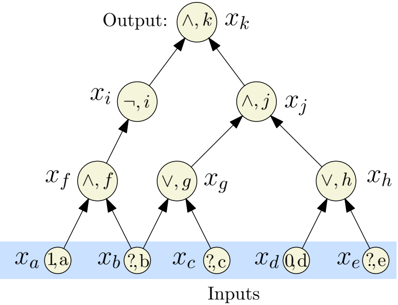
Converting a circuit into a CNF formula

Convert each sub-formula to an equivalent CNF formula

x_k	x_k
$x_k = x_i \wedge x_j$	$(\neg x_k \vee x_i) \wedge (\neg x_k \vee x_j) \wedge (x_k \vee \neg x_i \vee \neg x_j)$
$x_j = x_g \wedge x_h$	$(\neg x_j \vee x_g) \wedge (\neg x_j \vee x_h) \wedge (x_j \vee \neg x_g \vee \neg x_h)$
$x_i = \neg x_f$	$(x_i \vee x_f) \wedge (\neg x_i \vee \neg x_f)$
$x_h = x_d \vee x_e$	$(x_h \vee \neg x_d) \wedge (x_h \vee \neg x_e) \wedge (\neg x_h \vee x_d \vee x_e)$
$x_g = x_b \vee x_c$	$(x_g \vee \neg x_b) \wedge (x_g \vee \neg x_c) \wedge (\neg x_g \vee x_b \vee x_c)$
$x_f = x_a \wedge x_b$	$(\neg x_f \vee x_a) \wedge (\neg x_f \vee x_b) \wedge (x_f \vee \neg x_a \vee \neg x_b)$
$x_d = 0$	$\neg x_d$
$x_a = 1$	x_a

Converting a circuit into a CNF formula

Take the conjunction of all the CNF sub-formulas



$$
\begin{aligned}
&x_k \wedge (\neg x_k \vee x_i) \wedge (\neg x_k \vee x_j) \\
&\wedge (x_k \vee \neg x_i \vee \neg x_j) \wedge (\neg x_j \vee x_g) \\
&\wedge (\neg x_j \vee x_h) \wedge (x_j \vee \neg x_g \vee \neg x_h) \\
&\wedge (x_i \vee x_f) \wedge (\neg x_i \vee \neg x_f) \\
&\wedge (x_h \vee \neg x_d) \wedge (x_h \vee \neg x_e) \\
&\wedge (\neg x_h \vee x_d \vee x_e) \wedge (x_g \vee \neg x_b) \\
&\wedge (x_g \vee \neg x_c) \wedge (\neg x_g \vee x_b \vee x_c) \\
&\wedge (\neg x_f \vee x_a) \wedge (\neg x_f \vee x_b) \\
&\wedge (x_f \vee \neg x_a \vee \neg x_b) \wedge (\neg x_d) \wedge x_a
\end{aligned}
$$

We got a CNF formula that is satisfiable if and only if the original circuit is satisfiable.

Reduction: CSAT $\leq_P$ SAT

1. For each gate (vertex) v in the circuit, create a variable x_v
2. Case $\neg$: v is labeled $\neg$ and has one incoming edge from u (so $x_v = \neg x_u$). In SAT formula generate, add clauses $(x_u \vee x_v)$, $(\neg x_u \vee \neg x_v)$. Observe that

$$x_v = \neg x_u \text{ is true} \quad \Longleftrightarrow \quad \begin{array}{l}(x_u \vee x_v)\\ (\neg x_u \vee \neg x_v)\end{array} \text{ both true.}$$

Reduction: CSAT $\leq_P$ SAT

Continued...

1. Case $\vee$: So $x_v = x_u \vee x_w$. In **SAT** formula generated, add clauses $(x_v \vee \neg x_u)$, $(x_v \vee \neg x_w)$, and $(\neg x_v \vee x_u \vee x_w)$. Again, observe that

$$\Big(x_v = x_u \vee x_w\Big) \text{ is true } \iff \begin{array}{l}(x_v \vee \neg x_u), \\ (x_v \vee \neg x_w), \\ (\neg x_v \vee x_u \vee x_w)\end{array} \text{ all true.}$$

Reduction: CSAT $\leq_P$ SAT

Continued...

1. Case $\wedge$: So $x_v = x_u \wedge x_w$. In **SAT** formula generated, add clauses $(\neg x_v \vee x_u)$, $(\neg x_v \vee x_w)$, and $(x_v \vee \neg x_u \vee \neg x_w)$. Again observe that

$$x_v = x_u \wedge x_w \text{ is true} \iff \begin{matrix} (\neg x_v \vee x_u), \\ (\neg x_v \vee x_w), \\ (x_v \vee \neg x_u \vee \neg x_w) \end{matrix} \quad \text{all true.}$$

Reduction: CSAT $\leq_P$ SAT

Continued...

1. If v is an input gate with a fixed value then we do the following. If $x_v = 1$ add clause x_v. If $x_v = 0$ add clause $\neg x_v$
2. Add the clause x_v where v is the variable for the output gate

Correctness of Reduction

Need to show circuit C is satisfiable iff φ_C is satisfiable

$\Rightarrow$ Consider a satisfying assignment a for C

1. Find values of all gates in C under a
2. Give value of gate v to variable x_v; call this assignment a'
3. a' satisfies φ_C (exercise)

$\Leftarrow$ Consider a satisfying assignment a for φ_C

1. Let a' be the restriction of a to only the input variables
2. Value of gate v under a' is the same as value of x_v in a
3. Thus, a' satisfies C

Part II

Reducing Problems to SAT and Circuit SAT

Power of SAT and CSAT

SAT and **CSAT** are meta-problems

Allow us to express/model problem using constraints. In essense they allow programming with constraints of certain restricted type.

Goal: examples to drive home the point

Reduce Directed Hamilton Path to SAT

Given directed graph $G = (V, E)$, does it have a Hamilton path?

Given G obtain CNF formula φ_G such that G has a Hamilton Path iff φ_G is satisfiable

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- What are variables?
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One approach: G has a Hamilton path iff there is a permutation of the n vertices such that for each i there is an edge from vertex in position i to vertex in position $(i + 1)$

How do we express permutations?

Reduction continued

Define variable $x(u,i)$ if vertex u in position i in the permutation. Total of n^2 variables where $n = |V|$.

Constraints?

- For each u, exactly one of $x(u,1), x(u,2), \ldots, x(u,n)$ should be true

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  - $\bigvee_{i=1}^{n} x(u, i)$ to ensure that $x(u, i)$ is 1 for at least one i
  - For $i \neq j$ we add constraint $\neg x(u, i) \vee \neg x(u, j)$ to ensure that we cannot choose both to be 1 for any pair.
  - For each u we have a total of $(1 + n(n-1)/2)$ constraints. Total of $n(1 + n(n-1)/2)$ over all vertices.
- $x(u, i)$ and $x(v, i+1)$ *implies* edge (u, v) in $E(G)$

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- $x(u, i)$ and $x(v, i+1)$ *implies* edge (u, v) in $E(G)$ $(x(u, i) \wedge x(v, i+1)) \Rightarrow z(u, v)$ where $z(u, v)$ is 1 if $(u, v) \in E$ otherwise 0 ($z(u, v)$ is a constant, not a variable but to help notation). Convert implication constraint to CNF.

Vertex Cover to CSAT

Given graph $G = (V, E)$ and integer k, does G have a vertex cover of size at most k?

Recall $S \subseteq V$ is a vertex cover if each edge (u, v) is covered by S, that means $u \in S$ or $v \in S$.

How do we reduce to CSAT/SAT? What are the variables?

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Constraints?

- For each edge $(u, v) \in E$ a constraint $(x_u \vee x_v)$. Total of $|E|$ constraints.
- $\sum_{u \in V} x_u \leq k$. Not a boolean constraint! How?

Vertex Cover to CSAT

Expressing $\sum_{u \in V} x_u \leq k$ as a *circuit*.

- Given inputs $x_u, u \in V$ can create an addition circuit that outputs the sum $\sum_u x_u$ as a $\lceil \log n \rceil$ bit binary number
- Given two r-bit binary inputs $y_1, y_2, \ldots, y_r$ and $z_1, z_2, \ldots, z_r$ one can develop a boolean circuit to compare which one is greater
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Combine with the constraints to cover edges to obtain a CSAT instance with input variables $x_u, u \in V$

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What was the reduction? Given G output graph H where H is obtained by adding a new vertex v to G and connecting it by edges to all the original vertices of G. Argue that G is 3-colorable iff H is 4-colorable.

Question: Is there a reduction from **4-Color** to **3-Color**?

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Yes! **4-Color** is in NP and **4-Color** reduces to **SAT**
SAT reduces to **3-SAT** and **3-SAT** reduces to **3-Color** and hence ...

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SAT reduces to **3-SAT** and **3-SAT** reduces to **3-Color** and hence ...

Exercise: Give an explicit reduction from **4-Color** to **SAT**

Cook-Levin Theorem

Theorem (Cook-Levin)

SAT *is* **NP-Complete**.

How did they prove it? And why **SAT** or **CSAT**?

Proof is in retrospect simple.

- Fix any non-deterministic TM M and string w
- Does M accept w in $p(|w|)$ steps where $p()$ is some fixed polynomial?
- Can express computation of M on w using a polynomial sized circuit (or CNF formula) due to expressive power of constraints and local computation of TMs
- Thus, can reduce an *arbitrary* NP problem (since it corresponds to some non-deterministic poly-time TM M) to **SAT**

Mathematical Programming

SAT, **CSAT** are boolean constraint satisfaction problems.

Other frameworks: constraints involving linear inequalities, convex functions, polynomials etc

Useful to know: Integer Linear Programming (ILP), Linear Programming (LP), Mixed Integer Linear Programming (MIP), Convex Programming

Commercial packages available. ILP, MIP are NP-Hard but many small to medium problems can be solved in practice. Powerful and expressive constraint involving numbers, not just booleans.

Linear Programming

Problem

Real variables $x_1, x_2, \ldots, x_n$. Solve

$$
\begin{array}{ll}
\text{maximize/minimize} & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad \text{for } i = 1 \ldots p \\
& \sum_{j=1}^{n} a_{ij} x_j = b_i \quad \text{for } i = p+1 \ldots q \\
& \sum_{j=1}^{n} a_{ij} x_j \geq b_i \quad \text{for } i = q+1 \ldots m
\end{array}
$$

Input is matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, column vector $b = (b_i) \in \mathbb{R}^m$, and row vector $c = (c_j) \in \mathbb{R}^n$

Constraints are linear equations and inequalities. Objective is a linear function

Integer Linear Programming

Problem

Integer variables $x_1, x_2, \ldots, x_n$. Solve

$$
\begin{array}{lll}
\text{maximize/minimize} & \sum_{j=1}^{n} c_j x_j & \\
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& \sum_{j=1}^{n} a_{ij} x_j = b_i & \text{for } i = p+1 \ldots q \\
& \sum_{j=1}^{n} a_{ij} x_j \geq b_i & \text{for } i = q+1 \ldots n \\
& x_i \in \mathbb{Z} & \text{for } i = 1 \text{ to } d
\end{array}
$$

Input is matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$, column vector $b = (b_i) \in \mathbb{R}^m$, and row vector $c = (c_j) \in \mathbb{R}^n$

Constraints are linear equations and inequalities. Objective is a linear function but variables need to take *integer* values

Convex Programming

Problem

Real variables $x_1, x_2, \ldots, x_n$. $x \in \mathbb{R}^n$ Solve

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq b_i \quad \text{for } i = 1 \ldots m \end{aligned}$$

$f, g_1, g_2, \ldots, g_m$ are convex functions

Mathematical Programming

- LP is a specical case of Convex Programming
- LP can be solved in polynomial time
- Convex programs can be solved arbitrarily well in polynomial time (exact solution is tricky because of irrational solutions)
- ILP and MIP are NP-Hard (decision versions are NP-Complete).

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Why is convex programming solvable?

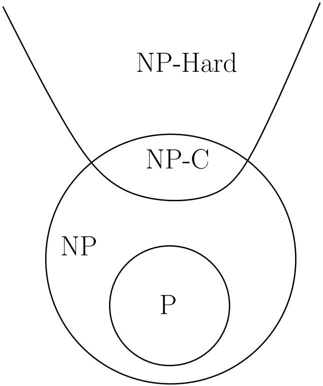
- For convex programs, local optimum is a global optimum!
- Local optimum can be found by local search! Gradient descent! Even for non-convex programs
- Gradient descent doesn't give a poly-time algorithm (gives a pseudo-polytime algorithm) but shows why efficiency is possible.

Interplay of Discrete and Continuous Optimization

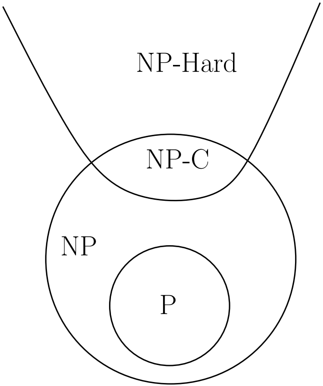
Both are fundamental and important and interplay has lot of impact!

- Machine learning: (deep) learning uses continuous optimization to train neural networks for classification and other discrete tasks
- Combinatorial optimization: use LP/SDP and other convex programming methods to solve combinatorial problems
- Scientific and numerical computing
- Statistics
- ...

Pictorial View



P and NP



P and NP

Possible scenarios:

1. P $=$ NP.
2. P $\neq$ NP

P and NP

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2. $\mathbf{P} \neq \mathbf{NP}$

Question: Suppose $\mathbf{P} \neq \mathbf{NP}$. Is every problem in $\mathbf{NP} \setminus \mathbf{P}$ also **NP-Complete**?

P and NP

Possible scenarios:

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2. $\mathsf{P} \neq \mathsf{NP}$

Question: Suppose $\mathsf{P} \neq \mathsf{NP}$. Is every problem in $\mathsf{NP} \setminus \mathsf{P}$ also **NP-Complete**?

Theorem (Ladner)

If $\mathsf{P} \neq \mathsf{NP}$ then there is a problem/language $X \in \mathsf{NP} \setminus \mathsf{P}$ such that X is not **NP-Complete**.

In fact a hierarcy of problems. However, no *natural* candidate.

The Big Picture