Graph Search

Lecture 15,16

March 9, 21, 2023

Part I

Graph Basics

Why Graphs?

1. Graphs have **many applications**!
2. Many important problems can be **reduced** to graph problems
3. **Fundamental object** in Computer Science
4. **Graph theory:** elegant, cfun and deep mathematics

Why Graphs?

1. Graphs have **many applications**!
2. Many important problems can be **reduced** to graph problems
3. **Fundamental object** in Computer Science
4. **Graph theory:** elegant, cfun and deep mathematics

In this course:

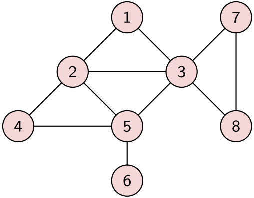
- Learn basic graph problems/algorithms and associated structure
    - Graph search, reachability and applications
    - Shortest paths and applications
    - Connection to dynamic programming
- Use graph reductions
- Learn about some basic hard graph problems

Undirected Graph

Definition

An undirected (simple) graph $G = (V, E)$ is a 2-tuple:

1. V is a set of vertices (also referred to as nodes/points)
2. E is a set of edges where each edge $e \in E$ is a set of the form $\{u, v\}$ with $u, v \in V$ and $u \neq v$.



Example

In figure, $G = (V, E)$ where $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 7\}, \{3, 8\}, \{4, 5\}, \{5, 6\}, \{7, 8\}\}$.

Example: Modeling Problems as Search

State Space Search

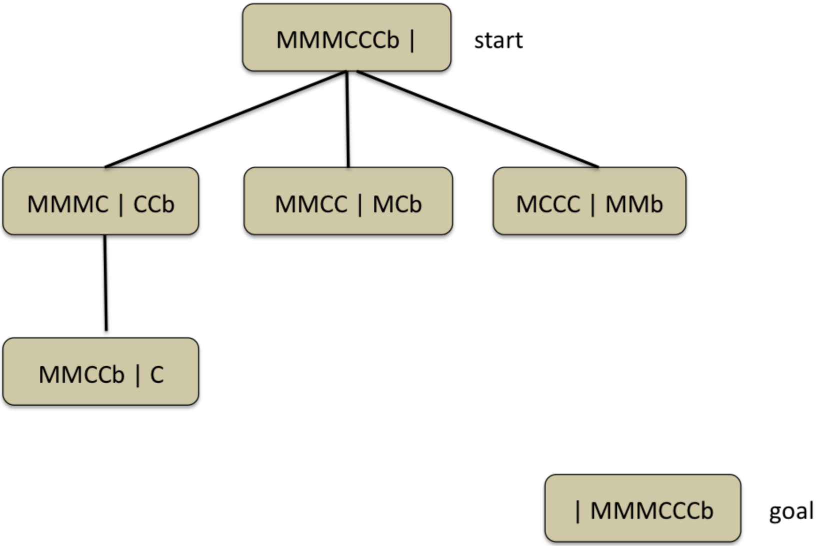
Many search problems can be modeled as search on a graph.
The trick is figuring out what the vertices and edges are.

Missionaries and Cannibals

- Three missionaries, three cannibals, one boat, one river
- Boat carries two people, must have at least one person
- Must all get across
- At no time can cannibals outnumber missionaries

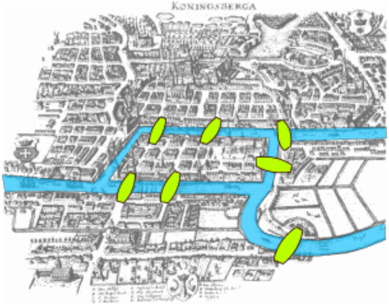
How is this a graph search problem?
What are the vertices?
What are the edges?

Example: Missionaries and Cannibals Graph



Bridges of Köningsberg

Figure from Wikipedia



Solved by Leonhard Euler (1707 — 1783). Start of graph theory.

Notation and Convention

Notation

An edge in an undirected graphs is an *unordered* pair of nodes and hence it is a set. Conventionally we use (u, v) for $\{u, v\}$ when it is clear from the context that the graph is undirected.

1. u and v are the end points of an edge $\{u, v\}$
2. Multi-graphs allow
    1. *loops* which are edges with the same node appearing as both end points
    2. *multi-edges*: different edges between same pairs of nodes
3. In this class we will assume that a graph is a simple graph unless explicitly stated otherwise.

Graph Representation I

Adjacency Matrix

Represent $G = (V, E)$ with n vertices and m edges using a $n \times n$ adjacency matrix A where

1. $A[i,j] = A[j,i] = 1$ if $\{i,j\} \in E$ and $A[i,j] = A[j,i] = 0$ if $\{i,j\} \notin E$.
2. **Advantage:** can check if $\{i,j\} \in E$ in $O(1)$ time
3. **Disadvantage:** needs $\Omega(n^2)$ space even when $m \ll n^2$

Graph Representation II

Adjacency Lists

Represent $G = (V, E)$ with n vertices and m edges using adjacency lists:

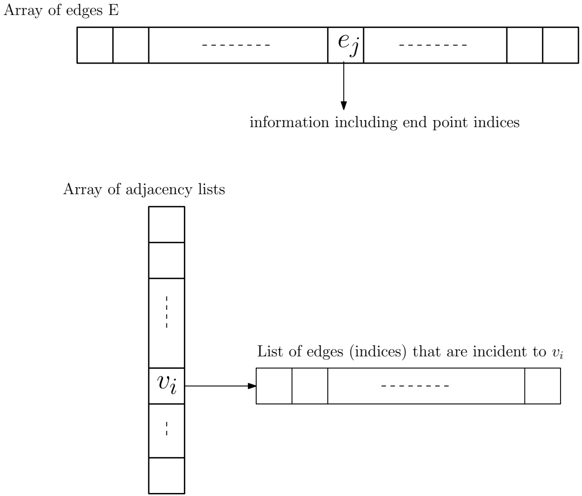
1. For each $u \in V$, $\text{Adj}(u) = \{v \mid \{u, v\} \in E\}$, the neighbors of u. Sometimes, $\text{Adj}(u)$ is the list of edges incident to u.
2. **Advantage:** space is $O(m + n)$
3. **Disadvantage:** cannot "easily" determine in $O(1)$ time whether $\{i, j\} \in E$
    1. By sorting each list, one can achieve $O(\log n)$ time
    2. By hashing "appropriately", one can achieve $O(1)$ time

Note: In this class we will assume that by default, graphs are represented using plain vanilla (unsorted) adjacency lists.

A Concrete Representation

- Assume vertices are numbered arbitrarily as $\{1, 2, \ldots, n\}$.
- Edges are numbered arbitrarily as $\{1, 2, \ldots, m\}$.
- Edges stored in an array/list of size m. $E[j]$ is j'th edge with info on end points which are integers in range 1 to n.
- Array Adj of size n for adjacency lists. $\text{Adj}[i]$ points to adjacency list of vertex i. $\text{Adj}[i]$ is a list of edge indices in range 1 to m.

A Concrete Representation



A Concrete Representation: Advantages

- Edges are explicitly represented/numbered. Scanning/processing all edges easy to do.
- Representation easily supports multigraphs including self-loops.
- Explicit numbering of vertices and edges allows use of arrays: $O(1)$-time operations are easy to understand.
- Can also implement via pointer based lists for certain dynamic graph settings.

Connectivity

Given a graph $G = (V, E)$:

1. A path is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq k-1$. The length of the path is $k-1$ (the number of edges in the path) and the path is from v_1 to v_k. Note: a single vertex u is a path of length 0.

Connectivity

Given a graph $G = (V, E)$:

1. A path is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq k-1$. The length of the path is $k-1$ (the number of edges in the path) and the path is from v_1 to v_k. Note: a single vertex u is a path of length 0.
2. A cycle is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ with $k \geq 3$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq k-1$ and $\{v_1, v_k\} \in E$. Single vertex or an edge are not a cycle according to this definition.
   Caveat: Some times people use the term cycle to also allow vertices to be repeated; we will use the term tour.

   Caveat: In multigraphs two parallel edges are considered a cycle. We stick to simple graphs.

Connectivity

Given a graph $G = (V, E)$:

1. A path is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \le i \le k-1$. The length of the path is $k-1$ (the number of edges in the path) and the path is from v_1 to v_k. Note: a single vertex u is a path of length 0.
2. A cycle is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ with $k \ge 3$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \le i \le k-1$ and $\{v_1, v_k\} \in E$. Single vertex or an edge are not a cycle according to this definition.

   Caveat: Some times people use the term cycle to also allow vertices to be repeated; we will use the term tour.

   Caveat: In multigraphs two parallel edges are considered a cycle. We stick to simple graphs.
3. A vertex u is connected to v if there is a path from u to v.

Connectivity

Given a graph $G = (V, E)$:

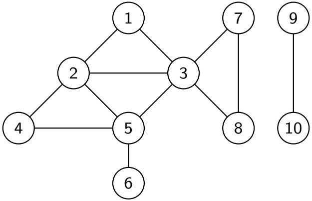
1. A path is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq k - 1$. The length of the path is $k - 1$ (the number of edges in the path) and the path is from v_1 to v_k. Note: a single vertex u is a path of length 0.
2. A cycle is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ with $k \geq 3$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq k - 1$ and $\{v_1, v_k\} \in E$. Single vertex or an edge are not a cycle according to this definition.
   Caveat: Some times people use the term cycle to also allow vertices to be repeated; we will use the term tour.

   Caveat: In multigraphs two parallel edges are considered a cycle. We stick to simple graphs.
3. A vertex u is connected to v if there is a path from u to v.
4. The connected component of u, $\text{con}(u)$, is the set of all vertices connected to u. Is $u \in \text{con}(u)$?

Connectivity contd

Define a relation C on $V \times V$ as uCv if u is connected to v

1. In undirected graphs, connectivity is a reflexive, symmetric, and transitive relation. Connected components are the equivalence classes.
2. Graph is connected if only one connected component.



Connectivity Problems

Algorithmic Problems

1. Given graph G and nodes u and v, is u *connected* to v?
2. Given G and node u, find all nodes that are connected to u.
3. Find all connected components of G.

Connectivity Problems

Algorithmic Problems

1. Given graph G and nodes u and v, is u *connected* to v?
2. Given G and node u, find all nodes that are connected to u.
3. Find all connected components of G.

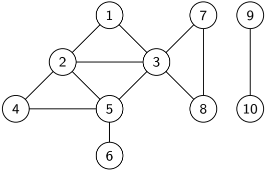
Can be accomplished in $O(m + n)$ time using **BFS** or **DFS**.
BFS and **DFS** are refinements of a basic search procedure which is good to understand on its own.

Basic Graph Search in Undirected Graphs

Given $G = (V, E)$ and vertex $u \in V$. Let $n = |V|$.

```
Explore(G, u):
    S ← {u}
    while (there is edge (u, v) with u ∈ S, v ∉ S) do
        S ← S ∪ {v}

    Output S
```

Example



Basic Graph Search in Undirected Graphs

Given $G = (V, E)$ and vertex $u \in V$. Let $n = |V|$.

```
Explore(G, u):
    S ← {u}
    while (there is edge (u, v) with u ∈ S, v ∉ S) do
        S ← S ∪ {v}

    Output S
```

Proposition

Explore(G, u) *terminates with* $S = con(u)$.

Proof?

Basic Graph Search in Undirected Graphs

Given $G = (V, E)$ and vertex $u \in V$. Let $n = |V|$.

```
Explore(G, u):
    S ← {u}
    while (there is edge (u, v) with u ∈ S, v ∉ S) do
        S ← S ∪ {v}

    Output S
```

Proposition

Explore(G, u) *terminates with* $S = con(u)$.

Proof? induction on loop invariant

- All vertices in S are connected to u (true at start)
- $|S|$ increases or loop is exited
- Loop terminates implies no edges leaving S

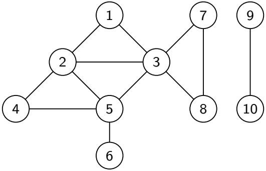
Basic Graph Search in Undirected Graphs

Given $G = (V, E)$ and vertex $u \in V$. Let $n = |V|$.

```
Explore(G,u):
    array Visited[1..n]
    Initialize:  Set Visited[i] = FALSE for 1 ≤ i ≤ n
    List:  ToExplore, S
    Add u to ToExplore and to S, Visited[u] = TRUE
    while (ToExplore is non-empty) do
        Remove node x from ToExplore
        for each edge (x, y) in Adj(x) do
            if (Visited[y] = FALSE)
                Visited[y] ← TRUE
                Add y to ToExplore
                Add y to S
    Output S
```

A concrete and fast implementation with simple marking and lists.

Example



Properties of Basic Search

Proposition

Explore(G, u) *terminates with* $S = con(u)$.

Properties of Basic Search

Proposition

Explore(G, u) *terminates with* $S = con(u)$.

Proof Sketch.

- Once Visited$[i]$ is set to TRUE it never changes. Hence a node is added only once to ToExplore. Thus algorithm terminates in at most n iterations of while loop.
- By induction on iterations, can show $v \in S \Rightarrow v \in \text{con}(u)$
- Since each node $v \in S$ was in ToExplore and was explored, no edges in G leave S. Hence no node in $V - S$ is in $\text{con}(u)$.
- Thus $S = \text{con}(u)$ at termination.

□

Properties of Basic Search

Proposition

Explore(G, u) *terminates in* $O(m + n)$ *time.*

Proof: easy exercise

Properties of Basic Search

Proposition

Explore(G, u) *terminates in* $O(m + n)$ *time.*

Proof: easy exercise

BFS and **DFS** are special case of BasicSearch.

1. Breadth First Search (**BFS**): use queue data structure to implementing the list ToExplore
2. Depth First Search (**DFS**): use stack data structure to implement the list ToExplore

Search Tree

One can create a natural search tree T rooted at u during search.

```
Explore(G,u):
    array Visited[1..n]
    Initialize:  Set Visited[i] = FALSE for 1 ≤ i ≤ n
    List:  ToExplore, S
    Add u to ToExplore and to S, Visited[u] = TRUE
    Make tree T with root as u
    while (ToExplore is non-empty) do
        Remove node x from ToExplore
        for each edge (x, y) in Adj(x) do
            if (Visited[y] == FALSE)
                Visited[y] = TRUE
                Add y to ToExplore
                Add y to S
                Add y to T with x as its parent
    Output S
```

T is a spanning tree of $\mathrm{con}(u)$ rooted at u

Finding all connected components

Exercise: Modify Basic Search to find all connected components of a given graph G in $O(m + n)$ time.

Part II

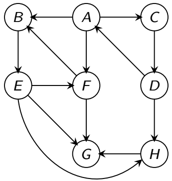
Directed Graphs and Decomposition

Directed Graphs

Definition

A directed graph $G = (V, E)$ consists of

1. set of vertices/nodes V and
2. a set of edges/arcs $E \subseteq V \times V$.



An edge is an *ordered* pair of vertices. (u, v) different from (v, u).

Examples of Directed Graphs

In many situations relationship between vertices is asymmetric:

1. Road networks with one-way streets.
2. Web-link graph: vertices are web-pages and there is an edge from page p to page p' if p has a link to p'. Web graphs used by Google with PageRank algorithm to rank pages.
3. Dependency graphs in variety of applications: link from x to y if y depends on x. Make files for compiling programs.
4. Program Analysis: functions/procedures are vertices and there is an edge from x to y if x calls y.

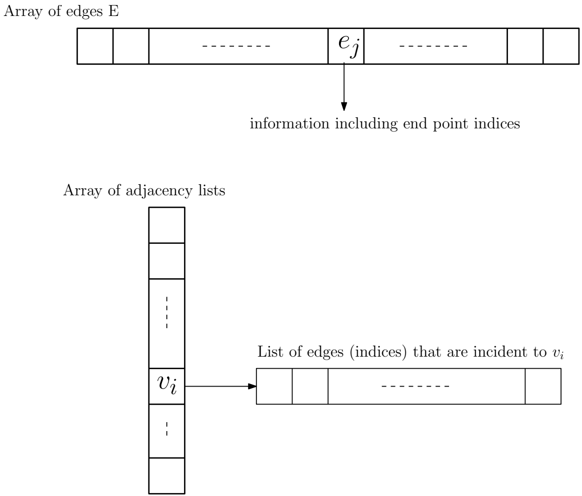
Directed Graph Representation

Graph $G = (V, E)$ with n vertices and m edges:

1. **Adjacency Matrix**: $n \times n$ *asymmetric* matrix A. $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ if $(u, v) \notin E$. $A[u, v]$ is not same as $A[v, u]$.
2. **Adjacency Lists**: for each node u, $\text{Out}(u)$ (also referred to as $\text{Adj}(u)$) and $\text{In}(u)$ store out-going edges and in-coming edges from u.

Default representation is adjacency lists.

A Concrete Representation for Dir Graphs



Default assumption: $\text{Adj}(\boldsymbol{u}) = \text{Out}(\boldsymbol{u})$

Directed Connectivity

Given a graph $G = (V, E)$:

1. A **(directed) path** is a sequence of *distinct* vertices $v_1, v_2, \dots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \le i \le k-1$. The length of the path is $k-1$ and the path is from v_1 to v_k. By convention, a single node u is a path of length 0.

Directed Connectivity

Given a graph $G = (V, E)$:

1. A **(directed) path** is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$. The length of the path is $k - 1$ and the path is from v_1 to v_k. By convention, a single node u is a path of length 0.
2. A **cycle** is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$ and $(v_k, v_1) \in E$. By convention, a single node u is not a cycle. Note: Unlike undirected graphs $k = 2$ is allowed so (u, v) together with (v, u) form a cycle (called a digon)

Directed Connectivity

Given a graph $G = (V, E)$:

1. A **(directed) path** is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$. The length of the path is $k - 1$ and the path is from v_1 to v_k. By convention, a single node u is a path of length 0.
2. A **cycle** is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$ and $(v_k, v_1) \in E$. By convention, a single node u is not a cycle.
   Note: Unlike undirected graphs $k = 2$ is allowed so (u, v) together with (v, u) form a cycle (called a digon)
3. A vertex u can reach v if there is a path from u to v. Alternatively v can be reached from u

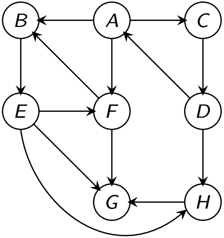
Directed Connectivity

Given a graph $G = (V, E)$:

1. A **(directed) path** is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k-1$. The length of the path is $k-1$ and the path is from v_1 to v_k. By convention, a single node u is a path of length 0.
2. A **cycle** is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k-1$ and $(v_k, v_1) \in E$. By convention, a single node u is not a cycle. Note: Unlike undirected graphs $k = 2$ is allowed so (u, v) together with (v, u) form a cycle (called a digon)
3. A vertex u can reach v if there is a path from u to v. Alternatively v can be reached from u
4. Let $\mathbf{rch}(u)$ be the set of all vertices reachable from u.

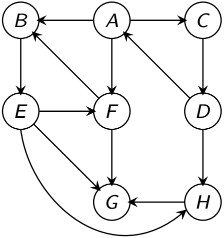
Connectivity contd

Asymmetricity: D can reach B but B cannot reach D



Connectivity contd

Asymmetricity: D can reach B but B cannot reach D



Questions:

1. Is there a notion of connected components?
2. How do we understand connectivity in directed graphs?

Strong Connected Components

Definition

Given a directed graph G, u is **strongly connected** to v if u can reach v *and* v can reach u; that is, $v \in \mathrm{rch}(u)$ and $u \in \mathrm{rch}(v)$.

Strong Connected Components

Definition

Given a directed graph G, u is **strongly connected** to v if u can reach v *and* v can reach u; that is, $v \in \text{rch}(u)$ and $u \in \text{rch}(v)$.

Define relation C where uCv if u is (strongly) connected to v.

Strong Connected Components

Definition

Given a directed graph G, u is **strongly connected** to v if u can reach v *and* v can reach u; that is, $v \in \mathrm{rch}(u)$ and $u \in \mathrm{rch}(v)$.

Define relation C where uCv if u is (strongly) connected to v.

Proposition

C is an equivalence relation, that is reflexive, symmetric and transitive.

Strong Connected Components

Definition

Given a directed graph G, u is **strongly connected** to v if u can reach v *and* v can reach u; that is, $v \in \text{rch}(u)$ and $u \in \text{rch}(v)$.

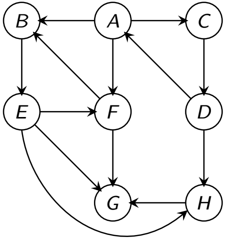
Define relation C where uCv if u is (strongly) connected to v.

Proposition

C is an equivalence relation, that is reflexive, symmetric and transitive.

Equivalence classes of C: *strong connected components* of G.
They *partition* the vertices of G.
$\text{SCC}(u)$: strongly connected component containing u.

Strongly Connected Components: Example



Directed Graph Connectivity Problems

1. Given G and nodes u and v, can u reach v?
2. Given G and u, compute $\text{rch}(u)$.
3. Given G and u, compute all v that can reach u, that is all v such that $u \in \text{rch}(v)$.
4. Find the strongly connected component containing node u, that is $\text{SCC}(u)$.
5. Is G strongly connected (a single strong component)?
6. Compute *all* strongly connected components of G.

Basic Graph Search in Directed Graphs

Given $G = (V, E)$ a directed graph and $u \in V$. Let $n = |V|$.

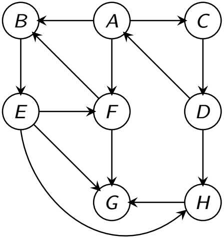
```
Explore(G,u):
    S ← {u}
    while (there is edge (u,v) with u ∈ S, v ∉ S) do
        S ← S ∪ {v}

    Output S
```

Basic Graph Search in Directed Graphs

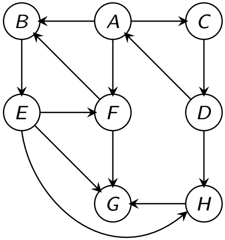
Given $G = (V, E)$ a directed graph and $u \in V$. Let $n = |V|$.

```
Explore(G,u):
    array Visited[1..n]
    Initialize:  Set Visited[i] = FALSE for 1 ≤ i ≤ n
    List:  ToExplore, S
    Add u to ToExplore and to S, Visited[u] = TRUE
    Make tree T with root as u
    while (ToExplore is non-empty) do
        Remove node x from ToExplore
        for each edge (x,y) in Adj(x) do
            if (Visited[y] == FALSE)
                Visited[y] = TRUE
                Add y to ToExplore
                Add y to S
                Add y to T with edge (x,y)
    Output S
```

Example



Example



Properties of Basic Search

Proposition

Explore(G, u) *terminates with* $S = rch(u)$.

Proof Sketch.

- Once Visited$[i]$ is set to TRUE it never changes. Hence a node is added only once to ToExplore. Thus algorithm terminates in at most n iterations of while loop.
- By induction on iterations, can show $v \in S \Rightarrow v \in \text{rch}(u)$
- Since each node $v \in S$ was in ToExplore and was explored, no edges in G leave S. Hence no node in $V - S$ is in rch(u). Caveat: In directed graphs edges can enter S.
- Thus $S = \text{rch}(u)$ at termination.

□

Properties of Basic Search

Proposition

Explore(G, u) *terminates in* $O(m + n)$ *time.*

Proposition

T *is a search tree rooted at* u *containing* S *with edges directed away from root to leaves.*

Proof: easy exercises

BFS and **DFS** are special case of Basic Search.

1. Breadth First Search (**BFS**): use queue data structure to implementing the list ToExplore
2. Depth First Search (**DFS**): use stack data structure to implement the list ToExplore

Exercise

Prove the following:

Proposition

Let $S = rch(u)$*. There is no edge* $(x, y) \in E$ *where* $x \in S$ *and* $y \notin S$*.*

Describe an example where $\text{rch}(u) \neq V$ and there are edges from $V \setminus \text{rch}(u)$ to $\text{rch}(u)$.

Directed Graph Connectivity Problems

1. Given G and nodes u and v, can u reach v?
2. Given G and u, compute $\mathrm{rch}(u)$.
3. Given G and u, compute all v that can reach u, that is all v such that $u \in \mathrm{rch}(v)$.
4. Find the strongly connected component containing node u, that is $\mathrm{SCC}(u)$.
5. Is G strongly connected (a single strong component)?
6. Compute *all* strongly connected components of G.

Directed Graph Connectivity Problems

1. Given G and nodes u and v, can u reach v?
2. Given G and u, compute $\text{rch}(u)$.
3. Given G and u, compute all v that can reach u, that is all v such that $u \in \text{rch}(v)$.
4. Find the strongly connected component containing node u, that is $\mathrm{SCC}(u)$.
5. Is G strongly connected (a single strong component)?
6. Compute *all* strongly connected components of G.

First five problems can be solved in $O(n + m)$ time by via Basic Search (or **BFS**/**DFS**). The last one can also be done in linear time but requires a rather clever **DFS** based algorithm.

Algorithms via Basic Search - I

1. Given G and nodes u and v, can u reach v?
2. Given G and u, compute $\mathrm{rch}(u)$.

Use $Explore(G, u)$ to compute $\mathrm{rch}(u)$ in $O(n + m)$ time.

Algorithms via Basic Search - II

1. Given G and u, compute all v that can reach u, that is all v such that $u \in \text{rch}(v)$.

Algorithms via Basic Search - II

1. Given G and u, compute all v that can reach u, that is all v such that $u \in \text{rch}(v)$.
   Naive: $O(n(n+m))$

Algorithms via Basic Search - II

1. Given G and u, compute all v that can reach u, that is all v such that $u \in \text{rch}(v)$.
   Naive: $O(n(n + m))$

Definition (Reverse graph.)

Given $G = (V, E)$, G^{rev} is the graph with edge directions reversed $G^{rev} = (V, E')$ where $E' = \{(y, x) \mid (x, y) \in E\}$

Algorithms via Basic Search - II

1. Given G and u, compute all v that can reach u, that is all v such that $u \in \text{rch}(v)$.
   Naive: $O(n(n+m))$

Definition (Reverse graph.)

Given $G = (V, E)$, G^{rev} is the graph with edge directions reversed $G^{rev} = (V, E')$ where $E' = \{(y, x) \mid (x, y) \in E\}$

Compute $\text{rch}(u)$ in G^{rev}!

1. **Correctness:** exercise
2. **Running time:** $O(n+m)$ to obtain G^{rev} from G and $O(n+m)$ time to compute $\text{rch}(u)$ via Basic Search. If both $Out(v)$ and $In(v)$ are available at each v then no need to explicitly compute G^{rev}. Can do $Explore(G, u)$ in G^{rev} implicitly.

Algorithms via Basic Search - III

$\mathrm{SCC}(G, u) = \{v \mid u \text{ is strongly connected to } v\}$

Algorithms via Basic Search - III

$\mathrm{SCC}(G, u) = \{v \mid u \text{ is strongly connected to } v\}$

1. Find the strongly connected component containing node u. That is, compute $\mathrm{SCC}(G, u)$.

Algorithms via Basic Search - III

$\mathrm{SCC}(G, u) = \{v \mid u \text{ is strongly connected to } v\}$

1. Find the strongly connected component containing node u. That is, compute $\mathrm{SCC}(G, u)$.

$\mathrm{SCC}(G, u) = \mathrm{rch}(G, u) \cap \mathrm{rch}(G^{rev}, u)$

Algorithms via Basic Search - III

$\mathrm{SCC}(G, u) = \{v \mid u \text{ is strongly connected to } v\}$

1. Find the strongly connected component containing node u. That is, compute $\mathrm{SCC}(G, u)$.

$\mathrm{SCC}(G, u) = \mathrm{rch}(G, u) \cap \mathrm{rch}(G^{rev}, u)$

Hence, $\mathrm{SCC}(G, u)$ can be computed with $Explore(G, u)$ and $Explore(G^{rev}, u)$. Total $O(n + m)$ time.

Why can $\mathrm{rch}(G, u) \cap \mathrm{rch}(G^{rev}, u)$ be done in $O(n)$ time?

Algorithms via Basic Search - IV

1. Is G strongly connected?

Algorithms via Basic Search - IV

1. Is G strongly connected?

Pick arbitrary vertex u. Check if $\mathrm{SCC}(G, u) = V$.

Algorithms via Basic Search - V

1. Find *all* strongly connected components of G.

Algorithms via Basic Search - V

1. Find *all* strongly connected components of G.

```
While G is not empty do
    Pick arbitrary node u
    find S = SCC(G, u)
    Remove S from G
```

Algorithms via Basic Search - V

1. Find *all* strongly connected components of G.

```
While G is not empty do
    Pick arbitrary node u
    find S = SCC(G, u)
    Remove S from G
```

Question: Why doesn't removing one strong connected components affect the other strong connected components?

Algorithms via Basic Search - V

1. Find *all* strongly connected components of G.

```
While G is not empty do
    Pick arbitrary node u
    find S = SCC(G, u)
    Remove S from G
```

Question: Why doesn't removing one strong connected components affect the other strong connected components?

Running time: $O(n(n+m))$.

Algorithms via Basic Search - V

1. Find *all* strongly connected components of G.

```
While G is not empty do
    Pick arbitrary node u
    find S = SCC(G, u)
    Remove S from G
```

Question: Why doesn't removing one strong connected components affect the other strong connected components?

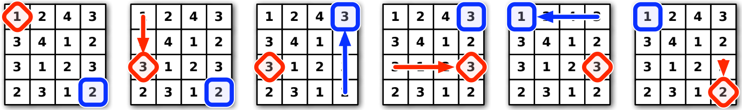
Running time: $O(n(n+m))$.

Question: Can we do it in $O(n+m)$ time?

Modeling Problems as Search

The following puzzle was invented by the infamous Mongolian puzzle-warrior Vidrach Itky Leda in the year 1473. The puzzle consists of an $n \times n$ grid of squares, where each square is labeled with a positive integer, and two tokens, one red and the other blue. The tokens always lie on distinct squares of the grid. The tokens start in the top left and bottom right corners of the grid; the goal of the puzzle is to swap the tokens.

In a single turn, you may move either token up, right, down, or left *by a distance determined by the **other** token*. For example, if the red token is on a square labeled 3, then you may move the blue token 3 steps up, 3 steps left, 3 steps right, or 3 steps down. However, you may not move a token off the grid or to the same square as the other token.



A five-move solution for a 4×4 Vidrach Itky Leda puzzle.

Describe and analyze an efficient algorithm that either returns the minimum number of moves required to solve a given Vidrach Itky Leda puzzle, or correctly reports that the puzzle has no solution. For example, given the puzzle above, your algorithm would return the number 5.

Undirected vs Directed Connectivity

Consider following problem.

- Given *undirected* graph $G = (V, E)$.
- Two subsets of nodes $R \subset V$ (red nodes) and $B \subset V$ (blue nodes). R and B non-empty.
- Describe linear-time algorithm to decide whether *every* red node can reach *every* blue node.

Undirected vs Directed Connectivity

Consider following problem.

- Given *undirected* graph $G = (V, E)$.
- Two subsets of nodes $R \subset V$ (red nodes) and $B \subset V$ (blue nodes). R and B non-empty.
- Describe linear-time algorithm to decide whether *every* red node can reach *every* blue node.

How does the problem differ in directed graphs?

Undirected vs Directed Connectivity

Consider following problem.

- Given *directed* graph $G = (V, E)$.
- Two subsets of nodes $R \subset V$ (red nodes) and $B \subset V$ (blue nodes).
- Describe linear-time algorithm to decide whether *every* red node can be reached by *some* blue node.