CS/ECE 374 Sec A ✧ Spring 2023

Homework 9

Due Wednesday, April 5, 2023 at 10am

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1. (a) Aaron and Alex decided to host a party for the CAs at Aaron's place. Yulie knew that Aaron likes empanadas and decided to buy them at one of the many places that sell them in Champaign (lucky for us!) on her way from her place to Aaron's place. Let $G = (V, E)$ be a directed graph with non-negative edge lengths $\ell(e), e \in E$ that represents the roads in town. Yulie entered her car to drive over and noticed that she may not have sufficient gas to drive all the way to Aaron's place. She estimated that her car can go D miles before her gas runs out. She wants to get to Aaron's place as fast as she can. Describe an efficient algorithm to help her accomplish the task of reaching Aaron's place with as little travel as feasible; she needs to buy empanadas but may or may not need to fill gas. Assume Yulie's house is at node s and Aaron's place is at node t and that the empanada shops are given by a set $X \subset V$ and the gas stations by a set $Y \subset V$. Assume that X, Y are disjoint sets. Also assume, for simplicity, that once Yulie fills gas she can travel an infinite distance. Note that Yulie could buy empanadas before or after filling up gas, as long as she does not run out of fuel on the way to the gas station. Express your running time as a function of n, the number of nodes, and m, the number of edges.

   (b) In the previous part we assumed that Yulie's car can go an infinite distance after filling up on gas. This is reasonable assumption for visits in the town. However, Yulie is now driving to visit a friend in a different state and has to fill up gas multiple times before reaching her destination. Assume that each full tank of gas lets her drive R miles. Describe an efficient algorithm that minimizes her driving distance. For simplicity assume that she starts with a full tank of gas.

   For both parts your algorithm should return ∞ if there is no way to reach the destination given the graph and the constraints.

2. Since you are taking an algorithms class you decided to create a fun Easter egg collection game. You set up a maze with one way streets that can be thought of as a directed graph $G = (V, E)$. Each node v in the maze has $w(v)$ amount of eggs located at v.

   (a) Each of your friends, starting at a given node s, has to figure out the maximum number of eggs they can collect. Note that eggs at node v can be collected only once even if the node v is visited again on the way to some other place.

   (b) Your friends complain that they can collect more eggs if they get to choose the starting node. You agree to their their request and ask them to maximize the number eggs they can collect starting at any node they choose.

   (c) Finally, you decide that it may be wise to restrict some of friends from collecting too many eggs, after all it is Easter. So you tell them that they can collect eggs from at most k locations where k is a parameter you give them.

Before you ask your friends to solve the game you need to know how to do it yourself! Describe efficient algorithms for these problems.

No proof necessary if you use reductions to standard algorithms via graph transformations and simple steps. Otherwise you need to prove the correctness.

3. **Not to submit:** Let $G = (V, E)$ be a directed graph with non-negative edge lengths $\ell(e), e \in E$. Dijkstra's algorithm can be used to find the shortest path tree rooted at any given node $s \in V$. In the standard shortest path problem the length of a path $v_1, v_2, \ldots, v_h$ is defined as $\sum_{i=1}^{h-1} \ell(v_i, v_{i+1})$ which is simply the sum of the lengths of the edges in the path. In various situations one needs different measures.

   (a) For a parameter $k \geq 1$ the k-norm length of a path $v_1, v_2, \ldots, v_k$ is defined to be $(\sum_{i=1}^{h-1} \ell(v_i, v_{i+1})^k)^{1/k}$. If $k = 1$ we get the standard length. Given G, $s, t \in V$ and $k \geq 1$ describe an algorithm to find the shortest k-norm length path from s to t in G. Give a small example where 2-norm s-t shortest path is different from the standard shortest path.

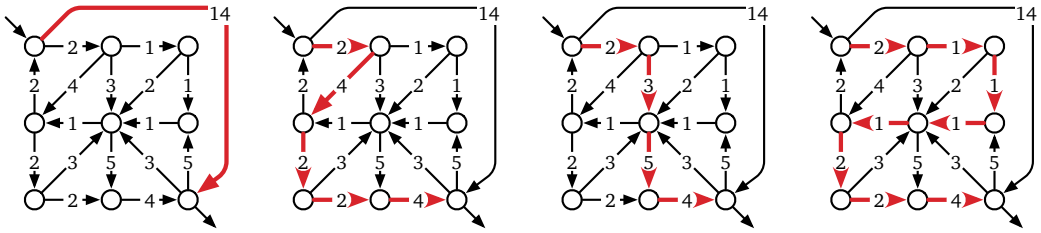
   (b) Consider the previous part but now suppose we set k to be a very large number. As $k \to \infty$ the k-norm of a path can be seen to be the maximum length of the edges in the path (assume that edge lengths are distinct). This corresponds to the ∞ norm of a vector which is the largest coordinate. In the context of paths, the length of the longest edge length in a path is called its *bottleneck* length. Describe an algorithm to compute the bottleneck shortest path distances from s to every node in G by adapting Dijkstra's algorithm.

   (c) We will consider an alternate algorithm to compute the s-t bottleneck shortest path distance. Given G, s,t and a value $\lambda \geq 0$, describe a reduction to s-t reachbility to decide whether there is a path from s-t with bottleneck length at most λ. Use this and binary search to find the s-t bottleneck distance.

   (d) Now consider another motivaton. Suppose each edge $e \in E$ has a probability $p(e)$ of failing. Given a path $v_1, v_2, \ldots, v_h$, what is the probability that none of the edges in the path fail assuming that the edges fail independently? Describe an algorithm to find the s-t path with the least probability of failing.

Solved Problem

4. Although we typically speak of "the" shortest path between two nodes, a single graph could contain several minimum-length paths with the same endpoints.



Four (of many) equal-length shortest paths.

Describe and analyze an algorithm to determine the *number* of shortest paths from a source vertex s to a target vertex t in an arbitrary directed graph G with weighted edges. You may assume that all edge weights are positive and that all necessary arithmetic operations can be performed in $O(1)$ time.

[Hint: Compute shortest path distances from s to every other vertex. Throw away all edges that cannot be part of a shortest path from s to another vertex. What's left?]

Solution: We start by computing shortest-path distances $dist(v)$ from s to v, for every vertex v, using Dijkstra's algorithm. Call an edge $u \to v$ ***tight*** if $dist(u) + w(u \to v) = dist(v)$. Every edge in a shortest path from s to t must be tight. Conversely, every path from s to t that uses only tight edges has total length $dist(t)$ and is therefore a shortest path!

Let H be the subgraph of all tight edges in G. We can easily construct H in $O(V + E)$ time. Because all edge weights are positive, H is a directed acyclic graph. It remains only to count the number of paths from s to t in H.

For any vertex v, let $PathsToT(v)$ denote the number of paths in H from v to t; we need to compute $PathsToT(s)$. This function satisfies the following simple recurrence:

$$PathsToT(v) = \begin{cases} 1 & \text{if } v = t \\ \displaystyle\sum_{v \to w} PathsToT(w) & \text{otherwise} \end{cases}$$

In particular, if v is a sink but $v \neq t$ (and thus there are no paths from v to t), this recurrence correctly gives us $PathsToT(v) = \sum \varnothing = 0$.

We can memoize this function into the graph itself, storing each value $PathsToT(v)$ at the corresponding vertex v. Since each subproblem depends only on its successors in H, we can compute $PathsToT(v)$ for all vertices v by considering the vertices in reverse topological order, or equivalently, by performing a depth-first search of H starting at s. The resulting algorithm runs in $O(V + E)$ time.

The overall running time of the algorithm is dominated by Dijkstra's algorithm in the preprocessing phase, which runs in $\boldsymbol{O(E \log V)}$ ***time***. ■

Rubric: 10 points = 5 points for reduction to counting paths in a dag + 5 points for the path-counting algorithm (standard dynamic programming rubric)