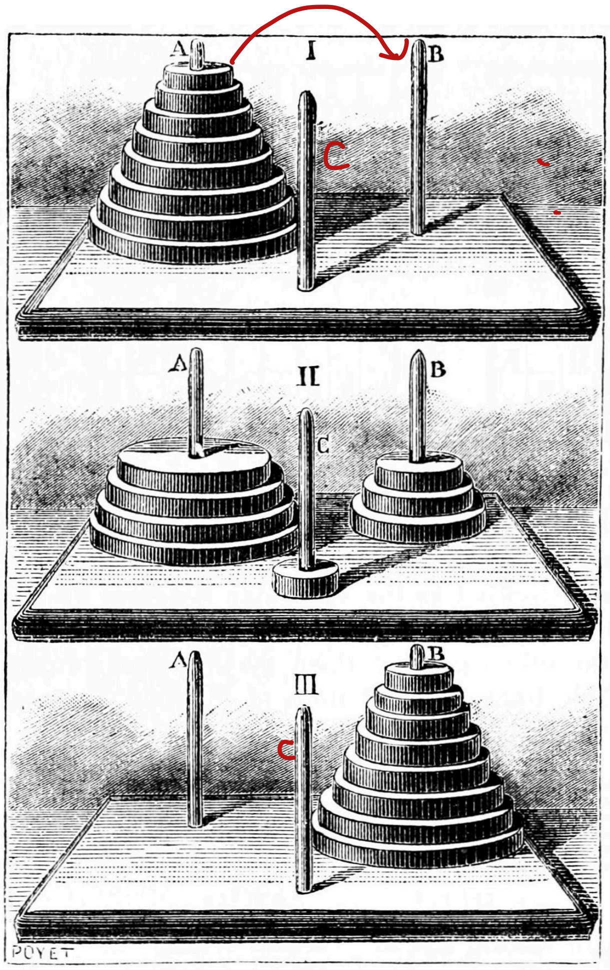
HW5 — Tue
GPS5 — Mon

MT1 + solns posted tomorrow AM

Every group submits own work

Algorithms: INDUCTION RECURSION = MAGIC



Lucas (1887)

Tower of Hanoi

Move smallest disk on one peg to another

Never move larger disk on top of smaller disk

Move n disks from A to B (via C)

if $n \geq 1$

→ Move $n-1$ disks from A to C (via B)

→ Move disk n from A to B

→ Move $n-1$ disks from B to C (via A)

Correctness by induction

Let $T(n)$ = # moves to solve n-disk Hanoi

$$T(n) = \begin{cases} 0 & \text{if } n = 0 \\ 2T(n-1)+1 & \text{otherwise} \end{cases}$$

n	0	1	2	3	4	5	6
$T(n)$	0	1	3	7	15	31	63

dynamic programming

$T(n) = 2^n - 1$?

Proof: $n=0$ $\quad 2^0 - 1 = 0$ ✓

$n \geq 1$

$$\begin{aligned} T(n) &= 2T(n-1)+1 \\ &= 2(2^{n-1}+1)+1 \quad \text{IH} \\ &= 2^n - 1 \quad \text{math} \end{aligned}$$

D

Input:	S	O	R	T	I	N	G	E	X	A	M	P	L
Divide:	S	O	R	T	I	N	G	E	X	A	M	P	L
Recurse Left:	I	N	O	R	S	T	G	E	X	A	M	P	L
Recurse Right:	I	N	O	R	S	T	A	E	G	L	M	P	X
Merge:	A	E	G	I	L	M	N	O	P	R	S	T	X

i j A^k

Merges $A[1..m]$ and $A[m+1..n]$ into $A[1..n]$

```
MERGE(A[1..n], m):
  i ← 1;  j ← m+1
  for k ← 1 to n
      if j > n
          B[k] ← A[i];  i ← i+1
      else if i > m
          B[k] ← A[j];  j ← j+1
      else if A[i] < A[j]
          B[k] ← A[i];  i ← i+1
      else
          B[k] ← A[j];  j ← j+1
  for k ← 1 to n
      A[k] ← B[k]
```

$O(n)$ time

Sorts $A[1..n]$ in-place

```
MERGESORT(A[1..n]):
  if n > 1
      m ← ⌊n/2⌋
      MERGESORT(A[1..m])        ⟨⟨Recurse!⟩⟩
      MERGESORT(A[m+1..n])      ⟨⟨Recurse!⟩⟩
      MERGE(A[1..n], m)
```

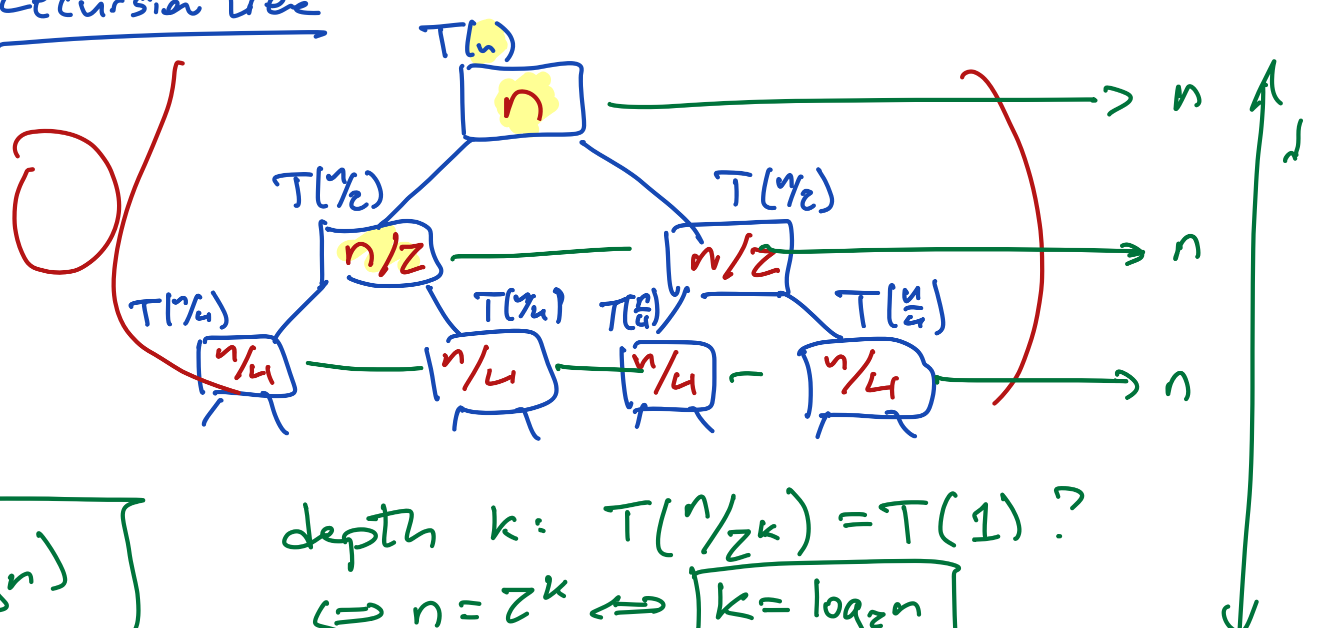
$$T(n) = T\left(\left\lfloor \tfrac{n}{2} \right\rfloor\right) + T\left(\left\lceil \tfrac{n}{2} \right\rceil\right) + O(n)$$

$$\boxed{T(n) = O(1) \text{ for all } n = O(1)}$$

Ignore floors + ceilings

$$\boxed{T(n) = 2T\left(\tfrac{n}{2}\right) + O(n)}$$

Recursion tree



$\boxed{O(n \cdot \log n)}$

depth k: $T(n/2^k) = T(1)$?

$\iff n = 2^k \iff \boxed{k = \log_2 n}$

Input:	S	O	R	T	I	N	G	E	X	A	M	P	L
Choose a pivot:	S	O	R	T	I	N	G	E	X	A	M	$\boxed{P}$	L
Partition:	A	G	O	E	I	N	L	M	$\boxed{P}$	T	X	S	R
Recurse Left:	A	E	G	I	L	M	N	O	$\boxed{P}$	T	X	S	R
Recurse Right:	A	E	G	I	L	M	N	O	$\boxed{P}$	R	S	T	X

$p = 12$

$r = 9$

```
PARTITION(A[1..n], p):
  swap A[p] ↔ A[n]
  ℓ ← 0                    ⟨⟨#items < pivot⟩⟩
  for i ← 1 to n − 1
      if A[i] < A[n]
          ℓ ← ℓ + 1
          swap A[ℓ] ↔ A[i]
  swap A[n] ↔ A[ℓ + 1]
  return ℓ + 1
```

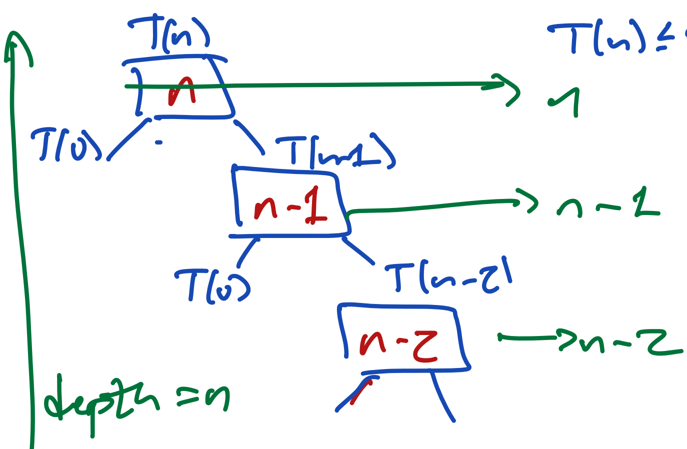
$O(n)$

```
QUICKSORT(A[1..n]):
  if (n > 1)
      Choose a pivot element A[p]
      r ← PARTITION(A, p)
      QUICKSORT(A[1..r − 1])     ⟨⟨Recurse!⟩⟩
      QUICKSORT(A[r + 1..n])     ⟨⟨Recurse!⟩⟩
```

$$T(n) \le O(n) + \max_r \big(T(r-1) + T(n-r)\big)$$

Guess $r = 1$ or $r = n$

$$T(n) \le O(n) + T(0) + T(n-1)$$



$$\boxed{T(n) = O(n^2)}$$

level k : $T(n-k) = T(0)$

$\Leftrightarrow$ $k = n$