A subsequence of a sequence (for example, an array, linked list, or string), obtained by removing zero or more elements and keeping the rest in the same sequence order. A subsequence is called a substring if its elements are contiguous in the original sequence. For example:

- SUBSEQUENCE, UBSEQU, and the empty string ε are all substrings (and therefore subsequences) of the string SUBSEQUENCE;
- SBSQNC, SQUEE, and EEE are all subsequences of SUBSEQUENCE but not substrings;
- QUEUE, EQUUS, and DIMAGGIO are not subsequences (and therefore not substrings) of SUBSEQUENCE.

Describe recursive backtracking algorithms for the following longest-subsequence problems. Don’t worry about running times.

1. Given an array \(A[1..n]\) of integers, compute the length of a longest increasing subsequence. A sequence \(B[1..\ell]\) is increasing if \(B[i] > B[i - 1]\) for every index \(i \geq 2\).

   For example, given the array
   \[
   \langle 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, 2, 7 \rangle
   \]
   your algorithm should return the integer 6, because \(\langle 1, 4, 5, 6, 8, 9 \rangle\) is a longest increasing subsequence (one of many).

2. Given an array \(A[1..n]\) of integers, compute the length of a longest decreasing subsequence. A sequence \(B[1..\ell]\) is decreasing if \(B[i] < B[i - 1]\) for every index \(i \geq 2\).

   For example, given the array
   \[
   \langle 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, 2, 7 \rangle
   \]
   your algorithm should return the integer 5, because \(\langle 9, 6, 5, 4, 2 \rangle\) is a longest decreasing subsequence (one of many).

3. Given an array \(A[1..n]\) of integers, compute the length of a longest alternating subsequence. A sequence \(B[1..\ell]\) is alternating if \(B[i] < B[i - 1]\) for every even index \(i \geq 2\), and \(B[i] > B[i - 1]\) for every odd index \(i \geq 3\).

   For example, given the array
   \[
   \langle 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, 2, 7 \rangle
   \]
   your algorithm should return the integer 17, because \(\langle 3, 1, 4, 1, 5, 2, 6, 5, 8, 7, 9, 3, 8, 4, 6, 2, 7 \rangle\) is a longest alternating subsequence (one of many).
Harder problems to think about later:


   For example, given the array

   $\langle 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, 2, 7 \rangle$

   your algorithm should return the integer 6, because $\langle 3, 1, 2, 5, 9 \rangle$ is a longest convex subsequence (one of many).

5. Given an array $A[1..n]$, compute the length of a longest palindrome subsequence of $A$. Recall that a sequence $B[1..\ell]$ is a palindrome if $B[i] = B[\ell - i + 1]$ for every index $i$.

   For example, given the array

   $\langle 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, 2, 7 \rangle$

   your algorithm should return the integer 7, because $\langle 4, 9, 5, 3, 5, 9, 4 \rangle$ is a longest palindrome subsequence (one of many).