Prove that each of the following languages is not regular, first using fooling sets and then (for problems 3, 4, and 5) using a reduction argument. You may use the fact (proven in class and in the lecture notes) that the language \( \{0^n 1^n \mid n \geq 0 \} \) is not regular. See the next page for a solved example showing both types of proof.

1. \( \{0^{2n} \mid n \geq 0 \} \)

2. \( \{0^{2n} 1^n \mid n \geq 0 \} \)

3. \( \{0^{m} 1^n \mid m \neq 2n \} \)
   
   [Hint: There is a short reduction argument, but write the fooling set argument first.]

4. Strings over \( \{0, 1\} \) where the number of 0s is exactly twice the number of 1s.
   
   [Hint: There is a short reduction argument, but write the fooling set argument first.]

5. Strings of properly nested parentheses (()), brackets [], and braces {}. For example, the string ([][)]{} is in this language, but the string ([]) is not, because the left and right delimiters don’t match.
   
   [Hint: There is a short reduction argument, but write the fooling set argument first.]

**Harder problems to think about later:**

6. Strings of the form \( w_1 \# w_2 \# \cdots \# w_n \) for some \( n \geq 2 \), where each substring \( w_i \) is a string in \( \{0, 1\}^* \), and some pair of substrings \( w_i \) and \( w_j \) are equal.

7. \( \{0^{n^2} \mid n \geq 0 \} \)

*8. \( \{w \in (\{0, 1\})^* \mid w \) is the binary representation of a perfect square\}
Solved problem:

9. Prove that the language \( L = \{ w \in (0 + 1)^* \mid \#(0, w) = \#(1, w) \} \) is not regular.

**Solution (fooling set \( 0^* \)):**

Consider the infinite set \( F = \{ 0^n \mid n \geq 0 \} \), or more simply \( F = 0^* \).

We claim that every pair of distinct strings in \( F \) has a distinguishing suffix.

Let \( x \) and \( y \) be arbitrary distinct strings in \( F \).

The definition of \( F \) implies \( x = 0^i \) and \( y = 0^j \) for some integers \( i \neq j \).

Let \( z \) be the string \( 1^i \).

Then \( xz = 0^i 1^i \in L \).

But \( yz = 0^j 1^i \notin L \), because \( i \neq j \).

So \( z \) is a distinguishing suffix for \( x \) and \( y \).

We conclude that \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.  

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This is exactly the proof from the lecture notes for the canonical non-regular language \( \{ 0^n 1^n \mid n \geq 0 \} \). The inner box is a proof that every pair of distinct strings in \( F \) has a distinguishing suffix.

**Solution (fooling set \( 0^* \)):**

For any natural number \( n \), let \( x_n = 0^n \), and let \( F = \{ x_n \mid n \geq 0 \} = 0^* \).

Let \( i \) and \( j \) be arbitrary distinct natural numbers.

Let \( z_{ij} \) be the string \( 1^i \).

Then \( x_i z_{ij} = 0^i 1^i \in L \).

But \( x_j z_{ij} = 0^j 1^i \notin L \), because \( i \neq j \).

So \( z_{ij} \) is a distinguishing suffix for \( x_i \) and \( x_j \).

We conclude that \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

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This is another way of writing exactly the same proof that emphasizes the counter intuition; any algorithm that recognizes \( L \) must count 0s.

**Solution (reduction via closure):** For the sake of argument, suppose \( L \) is regular.

Then the language \( L \cap 0^n 1^n = \{ 0^n 1^n \mid n \geq 0 \} \) would also be regular, because regular languages are closed under intersection.

But we proved in class that \( \{ 0^n 1^n \mid n \geq 0 \} \) is not regular; we’ve reached a contradiction.

We conclude that \( L \) cannot be regular.

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And this is why the proof for \( \{ 0^n 1^n \mid n \geq 0 \} \) also works verbatim for this language.