Homework 3
Due Tuesday, September 12, 2023 at 9pm Central Time

- Please start your solution to each lettered subproblem ((a), (b), (c), etc.) on a new page. Please also remember to tell Gradescope which page(s) are relevant for which subproblems.

1. Prove that the following languages over the alphabet $\Sigma = \{0, 1\}$ are not regular.
   (a) $\{0^a 1^b 1^c \mid 2b = a + c\}$.
   (b) The set of all palindromes in $\Sigma^*$ whose lengths are divisible by 7.
   (c) $\{1^m 0^n \mid m + n > 0 \text{ and } \gcd(m, n) = 1\}$
      Here $\gcd(m, n)$ denotes the greatest common divisor of $m$ and $n$: the largest integer $d$ such that both $m/d$ and $n/d$ are integers. In particular, $\gcd(1, n) = 1$ and $\gcd(0, n) = n$ for every positive integer $n$.

2. For each of the following languages over the alphabet $\Sigma = \{0, 1\}$, either prove that the language is regular (by constructing an appropriate DFA, NFA, or regular expression) or prove that the language is not regular (by constructing an infinite fooling set). Recall that $\Sigma^+$ denotes the set of all nonempty strings over $\Sigma$.
   (a) Strings in which the substrings 01 and 10 appear the same number of times. For example, $110011 \in L$ because both substrings appear once, but $0100011 \not\in L$.
   (b) Strings in which the substrings 00 and 11 appear the same number of times. For example, $110011 \in L$ because both substrings appear twice, but $0100011 \not\in L$.
   (c) $\{x y y x \mid x, y \in \Sigma^+\}$
   (d) $\{x y y z \mid x, y, z \in \Sigma^+\}$

   [Hint: Exactly two of these languages are regular.]
3. Practice only. Do not submit solutions.

A Moore machine is a variant of a finite-state automaton that produces output; Moore machines are sometimes called finite-state transducers. For purposes of this problem, a Moore machine formally consists of six components:

- A finite set $\Sigma$ called the input alphabet
- A finite set $\Gamma$ called the output alphabet
- A finite set $Q$ whose elements are called states
- A start state $s \in Q$
- A transition function $\delta : Q \times \Sigma \rightarrow Q$
- An output function $\omega : Q \rightarrow \Gamma$

More intuitively, a Moore machine is a graph with a special start vertex, where every node (state) has one outgoing edge labeled with each symbol from the input alphabet, and each node (state) is additionally labeled with a symbol from the output alphabet.

The Moore machine reads an input string $w \in \Sigma^*$ one symbol at a time. For each symbol, the machine changes its state according to the transition function $\delta$, and then outputs the symbol $\omega(q)$, where $q$ is the new state. Formally, we recursively define a transducer function $\omega^* : Q \times \Sigma^* \rightarrow \Gamma^*$ as follows:

$$\omega^*(q, w) = \begin{cases} 
\varepsilon & \text{if } w = \varepsilon \\
\omega(\delta(q, a)) \cdot \omega^*(\delta(q, a), x) & \text{if } w = ax 
\end{cases}$$

Given input string $w \in \Sigma^*$, the machine outputs the string $\omega^*(w, s) \in \Gamma^*$. The output language $L^\circ(M)$ of a Moore machine $M$ is the set of all strings that the machine can output:

$$L^\circ(M) := \{ \omega^*(s, w) \mid w \in \Sigma^* \}$$

(a) Let $M$ be an arbitrary Moore machine. Prove that $L^\circ(M)$ is a regular language.

(b) Let $M$ be an arbitrary Moore machine whose input alphabet $\Sigma$ and output alphabet $\Gamma$ are identical. Prove that the language

$$L^= (M) = \{ w \in \Sigma^* \mid w = \omega^*(s, w) \}$$

is regular. $L^= (M)$ consists of all strings $w$ such that $M$ outputs $w$ when given input $w$; these are also called fixed points for the transducer function $\omega^*$.

[Hint: These problems are easier than they look!]
Solved problems

4. For each of the following languages, either prove that the language is regular (by constructing an appropriate DFA, NFA, or regular expression) or prove that the language is not regular (by constructing an infinite fooling set).

Recall that a palindrome is a string that equals its own reversal: \( w = w^R \). Every string of length 0 or 1 is a palindrome.

(a) Strings in \((\{0, 1\})^*\) in which no prefix of length at least 2 is a palindrome.

Solution: Regular: \( \varepsilon + 01^* + 10^* \). Call this language \( L_a \).

Let \( w \) be an arbitrary non-empty string in \((\{0, 1\})^*\). Without loss of generality, assume \( w = \varepsilon x \) for some string \( x \). There are two cases to consider.

- If \( x \) contains a 0, then we can write \( w = 01^n0y \) for some integer \( n \) and some string \( y \). The prefix \( 01^n0 \) is a palindrome of length at least 2. Thus, \( w \notin L_a \).

- Otherwise, \( x \in 1^* \). Every non-empty prefix of \( w \) is equal to \( 01^n0 \) for some non-negative integer \( n \leq |x| \). Every palindrome that starts with 0 also ends with 0, so the only palindrome prefixes of \( w \) are \( \varepsilon \) and \( 0 \), both of which have length less than 2. Thus, \( w \in L_a \).

We conclude that \( \varepsilon x \in L_a \) if and only if \( x \in 1^* \). A similar argument implies that \( 1x \in L_a \) if and only if \( x \in 0^* \). Finally, trivially, \( \varepsilon \in L_a \). ■

Rubric: 2½ points = ½ for “regular” + 1 for regular expression + 1 for justification. This is more detail than necessary for full credit.

(b) Strings in \((\{0, 1, 2\})^*\) in which no prefix of length at least 2 is a palindrome.

Solution: Not regular. Call this language \( L_b \).

Consider the set \( F = \{012\}^+ \).

Let \( x \) and \( y \) be arbitrary distinct strings in \( F \).

Then \( x = (012)^i \) and \( y = (012)^j \) for some positive integers \( i \neq j \).

Without loss of generality, assume \( i < j \).

Let \( z \) be the suffix \( (210)^j \).

- \( xz = (012)^i(210)^j \) is a palindrome of length \( 6i \geq 2 \), so \( xz \notin L_b \).

- \( yz = (012)^i(210)^j \) has no palindrome prefixes except \( \varepsilon \) and \( 0 \), because \( i < j \), so \( yz \in L_b \).

Thus, \( z \) is a distinguishing suffix for \( x \) and \( y \).

We conclude that \( F \) is a fooling set for \( L_b \).

Because \( F \) is infinite, \( L_b \) cannot be regular. ■

Rubric: 2½ points = ½ for “not regular” + 2 for fooling set proof (standard rubric, scaled).
(c) Strings in $\{0, 1\}^*$ in which no prefix of length at least 3 is a palindrome.

**Solution:** Not regular. Call this language $L_c$.

Consider the set $F = (001101)^+$. Let $x$ and $y$ be arbitrary distinct strings in $F$. Then $x = (001101)^i$ and $y = (001101)^j$ for some positive integers $i \neq j$. Without loss of generality, assume $i < j$.

Let $z$ be the suffix $(101100)^i$.

- $xz = (001101)^i(101100)^i$ is a palindrome of length $12i \geq 2$, so $xz \notin L_b$.
- $yz = (001101)^j(101100)^i$ has no palindrome prefixes except $\epsilon$ and $0$ and $\epsilon$, because $i < j$, so $yz \in L_b$.

Thus, $z$ is a distinguishing suffix for $x$ and $y$. We conclude that $F$ is a fooling set for $L_c$. Because $F$ is infinite, $L_c$ cannot be regular.

**Rubric:** 2½ points = ½ for “not regular” + 2 for fooling set proof (standard rubric, scaled).

(d) Strings in $\{0, 1\}^*$ in which no substring of length at least 3 is a palindrome.

**Solution:** Regular. Call this language $L_d$.

Every palindrome of length at least 3 contains a palindrome substring of length 3 or 4. Thus, the complement language $\overline{L_d}$ is described by the regular expression

$$(0 + 1)^* (\epsilon + 0 + 1 + 00 + 01 + 10 + 11 + 001 + 011 + 100 + 110 + 0011 + 1100) (0 + 1)^*$$

Thus, $\overline{L_d}$ is regular, so its complement $L_d$ is also regular.

**Solution:** Regular. Call this language $L_d$.

In fact, $L_d$ is finite! Appending either 0 or 1 to any of the underlined strings creates a palindrome suffix of length 3 or 4.

$$\epsilon + 0 + 1 + 00 + 01 + 10 + 11 + 001 + 011 + 100 + 110 + 0011 + 1100$$

**Rubric:** 2½ points = ½ for “regular” + 2 for proof:

- 1 for expression for $\overline{L_d}$
- 1 for applying closure
- 1 for regular expression
- 1 for justification