Final Exam: May 12, 8-11 am. Online. Proctored. Last 30 mins for second upload. So essentially you will have ~150 mins to work on the questions.

Cumulative: Roughly will include
~ 1 greedy que.
~ 1 NP-completeness que
short questions on undecidability.

* Greedy

MT2 Fodder, #19: Given a set of intervals
\([a_1, b_1], [a_2, b_2], \ldots, [a_m, b_m]\), find the smallest set of points that "stabs" each interval at least once.

Eg.

```
  o
  
  P_1
  
  P_2
  
  P = \{P_1, P_2\}
```

GreedyAlg'm:

1. \(P = \emptyset, I = \{1, \ldots, n\}\)
2. repeat 3
3. pick \(i \in I\) or smallest \(b_i\)
4. \(P \leftarrow P \cup \{i\}\)
   
   ```
   ```
1. \( i \leftarrow 1 \) if \( i \) is some interval \( J_k \) that is not covered by \( J_{k-1} \) or \( J_k \).

2. \( p_n \leftarrow J_n \) if \( i \leq \arg \min_j b_j \).

3. \( p^{*} = \bigcup_{i} J_i \) if \( i \leq \arg \min_j b_j \).

4. Remove all intervals stabbed by \( b_i \) from \( I \).

5. Until \( I = \emptyset \)

**Running time:** \( O(n \log n + n) \) like interval scheduling.

**Correctness pf:** Let \( p^{*} \) be an optimal solution.

Let \( p_{i} \in p^{*} \) be the leftmost/min point in \( p^{*} \)

\[ p^{*} \]

Let \( b_{i} \in p^{*} \) be the first point picked by greedy.

\[ a_{1} \quad b_{1} \]

\[ a_{1} \quad b_{1} \]

**Claim:** If \( [a_{i}, b_{i}] \) is stabbed by \( p_{i} \), then it is also stabbed by \( b_{i} \).

**pf:** By choice of \( b_{i} = \arg \min \) \( b_j \), we know that \( a_{i} \leq p \leq b_{1} \leq b_{i} \),
\[ a_i \leq b_i \leq b_i \]
\[ \downarrow \]
\[ b_i \text{ stab } [a_i, b_i]. \]

\[ \exists \, \text{ feasible } \& \text{ optimal sol'} \]

\[ p^* = p^* \setminus \{p_i^*\} \cup \{b_i^*\}. \]

Repeat. \[ \Rightarrow \] By induction I am optimal sol' that exactly matches with the greedy sol'.

**NP-completeness:**

#4. Intra-HC:

input: undirected graph \( G = (V, E) \)

output: \( \text{YES } \iff \) a closed walk \( C \) that visits every vertex exactly once, except one vertex that it may not visit.

\[ \text{e.g.} \]

\[ \text{(a).} \]

\[ \text{(b). Intra-HC is } \text{NP-complete.} \]

\[ \text{... is in } \text{NP.} \]

\[ \subseteq \text{poly-size} \]

\[ \text{... } \text{in } \text{NP.} \]

\[ \forall \text{ } n, n_i = C \]
(b) \textbf{Intra-HC} is \textit{NP-wr}.

1. \textbf{Intra-HC} is in \textit{NP}.
   - Certificate: list of vertices $v_1, v_2, \ldots, v_n, v_i = C$

2. $C$ is a closed walk
   - Every vertex appears exactly once except one.

\textit{Poly-time:}

2. \textbf{HC} $\leq_p$ \textbf{Intra-HC}.

Given \textit{I/P} to \textit{HC}, cycle: $G = (V,E)$ undirected.

Conduct \textit{I/P} to \textbf{Intra-HC}: new graph $G' = (V', E')$ undirected.

\[ V' = V \cup \{v_1, v_2, \ldots, v_n\} \]

\[ E' = E \]

\textbf{Correctness:} \textbf{HC} in $G \iff \text{Intra-HC in } G'$

$\Rightarrow$ let $C = v_1, v_2, \ldots, v_n, v_i$ be a HC in $G$.

Then $C' = C$ is an \textbf{Intra-HC} in $G'$ because it covers all vertices $V'$ except vertex exactly once.

\[ \therefore \text{A } C' \text{ be intra-HC in } G' \]
(≤) Let $C'$ be intra-HC in $G'$.

Assuming $|V| \geq 2$, it must be that vertex $x$ is missing in $C'$.

Hence $C = C'$ is a HC in $G$.

**Ultra-HC:**

*Input:* Graph $G = (V, E)$ undirected.

*Output:* YES if $G$ contains a closed walk that contains every vertex exactly once except one vertex that can be repeated.

**Example:**

$e.d.a.b.c.a.e$
Construction:

\[ V' = V \cup \{ a', s, x^{\exists} \} \]
\[ E' = E \cup \{ as, sa', s \times^{\exists} \} \cup \{ a'x_{e} \mid x_{e} \in E^{3} \} \]

Connectivity:

\[ \text{IHC in } G \iff \exists \text{Ultra-HC in } G' \]
Connectivity: \[ \text{IHC in } G \iff \text{ Ultra-IHC in } G' \]

\[(\Rightarrow)\] Let \( C \) be a HC.

\[ C = \cdots u \sim a \sim u_2 \cdots \]

\[ \cdots u \sim a \sim s \sim a' \sim u_2 \cdots \text{ is an ultra-HC in } G'. \]

\[(\Leftarrow)\] Let \( C' \) be a ultra-HC in \( G' \). Then \( s \) must be repeated in \( C' \).

So \( C' = \cdots u_2 \sim a \sim s \sim s \sim a' \sim u_3 \cdots \)

\( C = \cdots u_2 \sim a \sim u_3 \cdots \)

is HC in \( G \).