More NP-Completeness:

Last Time: Independent Set is NP-complete

3SAT ≤p IS.

Cor: 1. VC is NPC (IS ≤p VC)
   2. SC is NPC (VC ≤p SC).

Hamiltonian cycle:

Input: undirected graph $G = (V, E)$
Output: YES iff $G$ has a cycle that visits each vertex exactly once.

$eg.$

Then: Dir-HC is NP-complete

PS (Karp's):

Directed version: Given directed graph $G$.

Dir-HC ≤p HC.

\[
\begin{array}{c}
\text{Dir-HC} \\
\text{HC}
\end{array}
\]

0. Dir-HC is in NP (exe).

\[v_{in} \rightarrow v \rightarrow v_{out}\]
1. Dir-HC is hard.

2. Vertex-cover \( \leq_p \) Dir-HC

Given an input to vertex cover, undir graph \( G = (V,E) \) iat \( k \),

"construct" input to Dir-HC: dir graph \( G' \) as follows: \( \forall e \in G \)

\[ \Rightarrow \text{construct line} \]

\[ (u,v) \in E \]

\[ u \rightarrow \]

\[ v \rightarrow \]

\[ \Rightarrow \text{add } z_1, \ldots, z_k \text{ vertices, edges to each } z_i \text{ from end of each } lv, \text{ and from each } z_i \text{ to start of each } lv. \]

\[ \Rightarrow \# \text{ vertices: } k + \# V, \quad \# \text{Edges: } O(nk) + O(m) \]

\[ \text{(construction is poly-time).} \]

\[ G = \]

\[ \text{Example:} \quad k = 2 \]

\[ \text{VC} = \{B, C\} \]

\[ \downarrow \text{construction is } G' \]
Correctness: A vertex cover $S$ in $G$ of size $\leq k$.

$\iff$ A Hamiltonian cycle in graph $G'$.

Proof sketch:

$(\Rightarrow)$ Let $S$ be a V.C. in $G$ and $|S| = k$.

Then for each $u \in S$, traverse $lu$ in cycle $C$ as follows: for each $(u,v) \in E$.

If $u \in S$, then $lu \longrightarrow lv$.

If $v \in S$, then $lu \longrightarrow lv$.

This constructs $k$ paths. Connect all of them through $21, \ldots, 2k$ into a single simple cycle.

$(\Leftarrow)$ Let $C$ be a Hamiltonian cycle in $G'$.

Construct a V.C. $S$ for $G$ as follows.

... $(u,v) \in E$, check how $C$ traverses...
Construct VC \( \subseteq \).
For each \((u,v) \in E\), check how \(C\) traverses its corresponding gadget.

\[ u \& s \implies \begin{cases} u \& S \\ v \& S \end{cases} \]

\[ \begin{cases} u \& S \\ v \& S \end{cases} \implies \begin{cases} u \& S \\ v \& S \end{cases} \]

It follows that \(S\) is a VC of \(G\) if \(S\) is checked.

But it is consistent.

\[ |S| = k \text{ because } C \text{ covers } z_1 \ldots z_k \text{ exactly once.} \]

* Subset-Sum:*

Input: Integers \(a_1, a_2, \ldots, a_n, W\) \(0 < a_i \leq W\)

Output: YES if \( \exists S \subseteq \{a_1, \ldots, a_n\} \text{ s.t. numbers in } S \text{ sums to } W. \)

*Eg.* \(10, 1, 35, 33, 5 \rightarrow W = 43 \rightarrow \text{YES.} \)

Note: Dynamic Program \(O(nW)\) time \(\text{(input size } \sim n \cdot \log W)\)

Not poly-time.

\(\text{Subset sum is NPC.}\)
Subset sum is NPC.

1. Subset sum in NP (exe)
2. Vertex cover \( \leq_p \) Subset sum.

Given i/p to VC: Graph \( G = (V, E) \), int \( k \)
Construct i/p to S.S.: integers \( a_1, \ldots, a_n \), \( W \)
as follows: let \( V = \{v_1, \ldots, v_n \} \), \( E = \{e_0, \ldots, e_{m-1} \} \)

\[
W = k \times 10^m + \sum_{j=0}^{m-1} a_j 10^j
\]

Poly-time:

\[
V, E, \quad a_i = 10^m + \sum_{j=0}^{m-1} a_j 10^j
\]

\[
E, V, \quad W = k \times 10^m + \sum_{j=0}^{m-1} 2 \cdot 10^j
\]

Given:\n
\[
G = \begin{array}{c}
  e_2 \\
  e_3 \\
  e_4 \\
  e_5 \\
  e_6 \\
\end{array}
\]

\[
\begin{array}{c c c c c c c c c c}
  e_5 & e_0 & (e_3) & e_2 & e_1 & e_0 \\
  v_6 & 1 & 0 & 0 & 0 & 0 & 1 & 1 &= a_1 \\
  v_7 & 1 & 0 & 0 & 1 & 1 & 0 & 1 &= a_2 \\
  v_3 & 1 & 1 & 1 & 0 & 1 & 1 & 0 &= a_3 \\
  v_4 & 1 & 0 & 1 & 1 & 0 & 0 & 0 &= a_4 \\
  v_5 & 1 & 1 & 0 & 0 & 0 & 0 & 0 &= a_5 \\
\end{array}
\]

(base 10)
(⇒) Given v.c. S

Construct $S' = \{a_i | \forall i \in S \cup \{b_i | \text{exactly one } \}$

Check $S'$ sums to $W$

(⇐) Given $S'$ sums to $W$

Construct v.c. $S$ as follows

$S = \{v_i | a_i \in S' \}$

$|S| = K$ because least digit in $W$ is $k$.

$4. S$ is a v.c. of $G$. Because for each edge $e_j = (u, v) \in E$, then an or at $S'$

0, w. jth digit = 0 if $W$ sum out

0, met