Tuesday, February 8, 2022 10:58 AM

Proving Non-Regularity

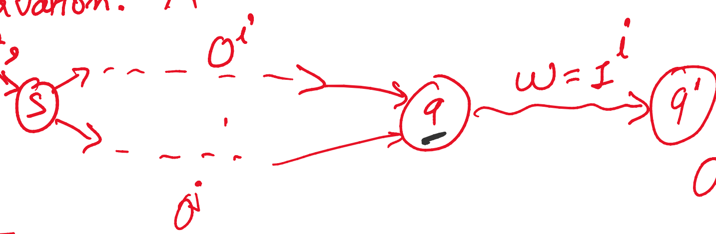
Warmup-Ex: $L = \{0^n 1^n \mid n \geq 0\}$ is not regular.

Pf: (By contradiction)

Suppose L is regular. Then L is accepted by some DFA $M = (Q, \Sigma, s, \delta, A)$

Observation: M has to "remember" # 0s seen.

If not, $i \neq j$



$0^i 1^i \in L \Rightarrow q' \in A$

Contradiction! X

$0^j 1^i \notin L \Rightarrow q' \notin A$

Claim: $\delta^*(s, 0^i) \neq \delta^*(s, 0^j)$, $\forall i \neq j$

Pf: (By contradiction)

Suppose, $\delta^*(s, 0^i) = \delta^*(s, 0^j)$

Then, $\delta^*(\delta^*(s, 0^i), 1^i) = \delta^*(\delta^*(s, 0^j), 1^i)$

$\Leftrightarrow \delta^*(s, 0^i 1^i) = \delta^*(s, 0^j 1^i)$

$\underset{A}{\Uparrow}$ $\qquad$ $\cancel{A}$ A $\qquad$ contradiction! $\square$

By claim, $\delta^*(s, 0^0), \delta^*(s, 0^1), \delta^*(s, 0^2), \ldots$

all are distinct states in Q.

Then $|Q|$ is infinite. Contradiction! $\blacksquare$

Rmk: The argument can be generalized.

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Definitions: Given a language $L \subseteq \Sigma^*$

$x, y \in \Sigma^*$, $x \neq y$ are distinguishable (w.r.t. L)

if $\exists w \in \Sigma^*$ s.t. $(xw \in L \;\&\; yw \notin L)$

OR $(xw \notin L \;\&\; yw \in L)$

A set $F \subseteq \Sigma^*$ is a distinguishing set ↳ fooling sets

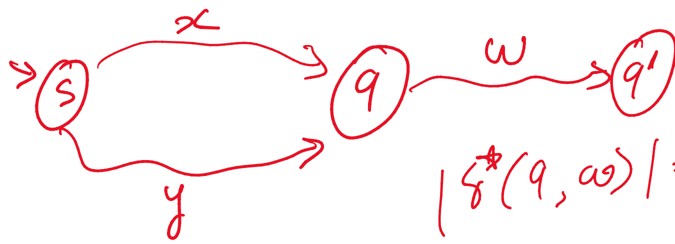
if any two string $x, y \in F$, $x \neq y$

x, y are distinguishable.

Observation: $x, y \in F$, $x \neq y$. Following DFA accepts L.

$xw \in L \Rightarrow q' \in A$

Q:



$yw \notin L \Rightarrow q' \notin A$

$|\delta^*(q, w)| = 1$

Thm: If L has an infinite sized fooling set F,

Then L is not regular

Pf: (By Contradiction)

Suppose L is regular

Then L is accepted by some DFA $M = (Q, \Sigma, s, \delta, A)$

Claim: $\forall x, y \in F$, $x \neq y$, $\delta^*(s, x) \neq \delta^*(s, y)$

Pf: Suppose, $\delta^*(s, x) = \delta^*(s, y)$

Since x, y distinguishable, $\exists w$

$(xw \in L \;\&\; yw \notin L)$ OR vice-versa.

Since x, y distinguishable, $\exists w$
s.t. $(xw \in L$ & $yw \notin L)$ OR vice-versa.

$$\delta^*(\delta^*(s, x), w) = \delta^*(\delta^*(s, y), w)$$

$$\Leftrightarrow \delta^*(s, xw) = \delta^*(s, yw)$$

$(\because xw \in L)$ $\Uparrow$ A $\qquad$ $\not A$ $(\because yw \notin L)$

A contradiction! $\square$

By claim, $\{\delta^*(s, x) \mid x \in F\}$ are all distinct states in Q

$\Rightarrow$ Q has infinite size. Contradiction! $\square$

Ex 1: $L = \{x \in \{0,1\}^* \mid \#0(x) = \#1(x)\}$ is not regular

Pf: Let $F = \{0^i \mid i \geq 0\}$

Given any two $x, y \in F$, $x \neq y$

$x = 0^i$, $y = 0^j$, $i \neq j$

Pick $w = 1^i$

$$\left.\begin{array}{l} xw = 0^i w \in L \\ yw = 0^j w \notin L \end{array}\right\} \Rightarrow x, y \text{ are distinguishable}$$

Then, F is a fooling set & has infinite size. $\square$

Ex 2: $L = \{$all palindrome in $\{0,1\}^*\}$ is not regular.

Pf: Let $F_1 = L$ X $\qquad$ $F_2 = \{10^i \mid i \geq 0\}$ X

$x, y \in F$ $\quad x \neq y$

Pick $w = x^R = x$

$x, y \in F$ $\quad x \neq y$

$x = 10^i$ $\quad y = 10^j$ X

Pick $w = 0^i 1$

$\left\{ \text{Pick } w = x^R = x \right.$

$x = 11 \quad y = 1111 \,;\, w = 11$

$xw = 1111 \in L$

$yw = 111111 \in L$

$x = 10$

Pick $w = 0^i 1$

$\{0^i \mid i \geq 0\}$

$x = 0^i,\ y = 0^j$

$w = 10^i$

$F = \{0^i 1 \mid i \geq 0\}$ $|F|$ is infinite.

Given any two $x, y \in F$, $x \neq y$

$x = 0^i 1$, $y = 0^j 1$, $i \neq j$

Pick, $w = 0^i$

$\left. \begin{array}{l} xw = 0^i 1 0^i \in L \\ yw = 0^j 1 0^i \notin L \end{array} \right\} \Rightarrow$ $x\ y$ are distinguishable

Then, F is an infinite sized fooling set for L ∎

Ex 3: $L = \{0^{n^2} \mid n \geq 0\}$ is not regular.

Pf: Let $F = \{0^i \mid i \geq 0\}$

Given $x, y \in F$, $x \neq y$

$x = 0^i$, $y = 0^j$ $\quad i \neq j$

$\underline{i < j \text{ w.l.o.g.}}$

Pick $w = 0^{j^2 - i} \neq \varepsilon$

$x\,y$ are distinguishable $\Leftarrow \left\{ \begin{array}{l} xw = 0^i \cdot 0^{j^2 - i} = 0^{j^2} \in L \\ yw = 0^j \cdot 0^{j^2 - i} = 0^{j^2 + j - i} \end{array} \right.$

$j^2 < j^2 + j - i < (j+1)^2$

$F = \{0^{i(i-1)} \mid i \geq 1\}$

$x = 0^{i(i-1)}$ $\quad y = 0^{j(j-1)}$

$i \neq j$

Pick $w = 0^i$

$xw = 0^{i^2 - i} \cdot 0^i = 0^{i^2} \in L$

$yw = 0^{j^2 - j} \cdot 0^i = 0^{\underline{j^2 - j + i}}$

How to show that $(j^2 - j + i) \neq k^2$ for any $k \geq 0$

any $k \ge 0$

$$j^2 < j^2 + j - i < (j+1)^2$$
$$j^2 + 2j + 1$$

Then F is a fooling set of infinite size □

Ex 4: $L = \{0^p \mid p \text{ is a prime}\}$ is not regular.

Pf: Let $F = \{0^i \mid i \ge 0\}$

Given any two $x, y \in F$, $x \neq y$

$x = 0^i$, $y = 0^j$, $i \neq j$
$i < j$ w.l.o.g.

Let k be the smallest integer s.t. $p + k(j-i)$ is composite (Note that, $1 \le k \le p$)

Pick $w = 0^{p+(k-1)(j-i)-i}$

Then $xw = 0^i \cdot 0^{p+(k-1)(j-i)-i} = 0^{\overbrace{p+(k-1)(j-i)}^{\text{prime by choice of } k}} \in L$

$yw = 0^j \cdot 0^{p+(k-1)(j-i)-i} = 0^{\overbrace{p+k(j-i)}^{\text{composite}}} \notin L$

x, y are distinguishable

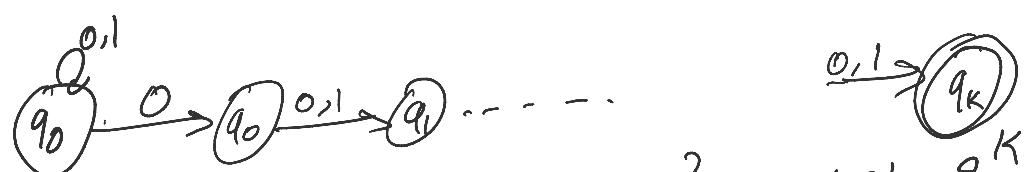
Then F is an infinite sized fooling set □

Extension: If F is a fooling set for language L. Then any DFA that accepts L must have $\ge |F|$ states.

Pf: (Same)!

EX: k constant, $L_k = \{x \in \{0,1\}^* \mid k\text{th symbol from right is } 0 \text{ in } x\}$

NFA



Let $F = \{x \in \{0,1\}^* \mid |x| = k\}$ $|F| = 2^k$.

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Given $x, y \in F$, $x \neq y$

$x = a_1 a_2 \dots a_k$ $\exists\, i \le k$ s.t.

$y = b_1 b_2 \dots b_k$ $a_i \neq b_i$ $a_i = 0,\ b_i = 1$ wlog.

$$xw = a_1 \dots \overset{0}{a_i} \dots a_k \overset{w}{\boxed{0^{i-1}}} \in L$$

$$yw = b_1 \dots \underbrace{\underset{1}{b_i} \dots b_k}_{k-i+1} \underbrace{\boxed{0^{i-1}}}_{i-1} \notin L$$

$\Big\}$ x, y are distinguishable.

F is a 2^k sized fooling set

$\Rightarrow$ any DFA for L requires $\geq 2^k$ states. ∎

Thm: [Myhill-Nerode]

Max fooling set size = min #states in a DFA.

$\underbrace{\text{fooling set size}}_{\#\text{ equivalence classes}}$

$x \equiv_L y$ if x, y are "not" distinguishable

$\uparrow$ equivalence relation

Σ^*