Prove that each of the following languages is not regular.

1. \( \{0^{2n} \mid n \geq 0\} \)

**Solution:**
Choose \( F = \{0^{2n} \mid n \geq 0\} \).
Let \( x \) and \( y \) be two arbitrary strings of \( F \) with \( x \neq y \).
Then \( x = 0^{2i} \) and \( y = 0^{2j} \) for some non-negative integers \( i \neq j \).
Choose \( z = 0^{2i} \).
Then \( xz = 0^{2i}0^{2i} = 0^{2i+1} \in L \).
And \( yz = 0^{2i}0^{2j} = 0^{2i+2j} \notin L \), because \( i \neq j \) (since \( 2^i + 2^j \) cannot be a power of 2).
Thus, \( F \) is a fooling set for \( L \).
Because \( F \) is infinite, \( L \) cannot be regular.

2. \( \{0^{2n}1^n \mid n \geq 0\} \)

**Solution:**
Choose \( F = \{0^i \mid i \geq 0\} \).
Let \( x \) and \( y \) be two arbitrary strings in \( F \) with \( x \neq y \).
Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).
Choose \( z = 0^i1^i \).
Then \( xz = 0^{2i}1^i \in L \).
And \( yz = 0^{i+j}1^i \notin L \), because \( i + j \neq 2i \).
Thus, \( F \) is a fooling set for \( L \).
Because \( F \) is infinite, \( L \) cannot be regular.

3. \( \{0^m1^n \mid m \neq 2n\} \)

**Solution:**
Choose \( F = \{0^i \mid i \geq 0\} \).
Let \( x \) and \( y \) be two arbitrary strings in \( F \) with \( x \neq y \).
Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).
Choose \( z = 0^i1^i \).
Then \( xz = 0^{2i}1^i \notin L \).
And \( yz = 0^{i+j}1^i \in L \), because \( i + j \neq 2i \).
Thus, \( F \) is a fooling set for \( L \).
Because \( F \) is infinite, \( L \) cannot be regular.
Strings over \{0, 1\} where the number of 0s is exactly twice the number of 1s.

Solution:
Choose \( F = \{0^i \mid i \geq 0\} \).
Let \( x \) and \( y \) be two arbitrary strings in \( F \) with \( x \neq y \).
Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).
Choose \( z = 0^i1^i \).
Then \( xz = 0^{2i}1^i \in L \).
And \( yz = 0^{i+j}1^i \not\in L \), because \( i + j \neq 2i \).
Thus, \( F \) is a fooling set for \( L \).
Because \( F \) is infinite, \( L \) cannot be regular.

Solution:
If \( L \) were regular, then the language

\[
((0 + 1)^* \setminus L) \cap 0^*1^* = \{0^m1^n \mid m \neq 2n\}
\]

would also be regular, because regular languages are closed under complement and intersection. But we just proved that \( \{0^m1^n \mid m \neq 2n\} \) is not regular in problem 3. [This proof would be worth full credit in homework or an exam, if we do not explicitly specify that you should use the fooling set method.]

Strings of properly nested parentheses \( () \), brackets \([] \), and braces \( \{\} \). For example, the string \( ([]\}) \) is in this language, but the string \( ([[]]) \) is not, because the left and right delimiters don’t match.

Solution:
Choose \( F = \{(^i \mid i \geq 0\} \).
Let \( x \) and \( y \) be two arbitrary strings in \( F \) with \( x \neq y \).
Then \( x = (^i \) and \( y = (^j \) for some non-negative integers \( i \neq j \).
Choose \( z = )^i \).
Then \( xz = (^i)^i \in L \).
And \( yz = (^j)^i \not\in L \), because \( i \neq j \).
Thus, \( F \) is a fooling set for \( L \).
Because \( F \) is infinite, \( L \) cannot be regular.

Strings of the form \( w_1\#w_2\#\cdots\#w_n \) for some \( n \geq 2 \), where each substring \( w_i \) is a string in \( \{0, 1\}^* \), and some pair of substrings \( w_i \) and \( w_j \) are equal.
Solution:
Choose \( F = \{0^i \mid i \geq 0\} \).
Let \( x \) and \( y \) be arbitrary strings in \( F \) with \( x \neq y \).
Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).
Choose \( z = \#0^i \).
Then \( xz = 0^i \#0^i \in L \).
And \( yz = 0^j \#0^i \notin L \), because \( i \neq j \).
Thus, \( F \) is a fooling set for \( L \).
Because \( F \) is infinite, \( L \) cannot be regular.

Extra problems

7 \( \{w \in (0 + 1)^* \mid w \text{ is the binary representation of a perfect square}\} \)

Solution:

Idea: We design our fooling set around numbers of the form \( (2^k + 1)^2 = 2^{2k} + 2^{k+1} + 1 \), which has binary representation \( 10^{k-2}10^{k+1} \). The argument is somewhat simpler if we further restrict \( k \) to be even.

Choose \( F = \{10^{2i}1 \mid i \geq 0\} \).
Let \( x \) and \( y \) be two distinct arbitrary strings in \( F \).
Then \( x = 10^{2i-1} \) and \( y = 10^{2j-1} \), for some positive integers \( i \neq j \). Without loss of generality, assume \( i < j \). (Otherwise, swap \( x \) and \( y \).)
Choose \( z = 0^{2i}1 \).
Then \( xz = 10^{2i-2}10^{2i+1} \) is the binary representation of \( 2^{4i} + 2^{2i+1} + 1 = (2^{2i} + 1)^2 \), and therefore \( xz \in L \).
On the other hand, \( yz = 10^{2j-2}10^{2i+1} \) is the binary representation of \( 2^{2i+2j} + 2^{2i+1} + 1 \). Simple algebra gives us the inequalities

\[
(2^{i+j})^2 = 2^{2i+2j} \\
< 2^{2i+2j} + 2^{2i+1} + 1 \\
< 2^{2(i+j)} + 2^{i+1+j+1} + 1 \\
= (2^{i+j} + 1)^2.
\]

So \( 2^{2i+2j} + 2^{2i+1} + 1 \) lies between two consecutive perfect squares, and thus is not a perfect square, which implies that \( yz \notin L \).
We conclude that \( F \) is a fooling set for \( L \). Because \( F \) is infinite, \( L \) cannot be regular.