1. Let \( L \) be an arbitrary regular language. Prove that the language \( \text{reverse}(L) := \{ w^R \mid w \in L \} \) is regular. 

*Hint:* Consider a DFA \( M \) that accepts \( L \) and construct a NFA that accepts \( \text{reverse}(L) \).

2. Let \( L \) be an arbitrary regular language. Prove that the language \( \text{insert1}(L) := \{ xy \mid xy \in L \} \) is regular. 

Intuitively, \( \text{insert1}(L) \) is the set of all strings that can be obtained from strings in \( L \) by inserting exactly one \( 1 \). For example, if \( L = \{ \varepsilon, OOK! \} \), then \( \text{insert1}(L) = \{ 1, 1OOK!, O1OK!, OO1K!, OOK1!, OOK!1 \} \).

Work on these later:

3. Let \( L \) be an arbitrary regular language. Prove that the language \( \text{delete1}(L) := \{ xy \mid x1y \in L \} \) is regular. 

Intuitively, \( \text{delete1}(L) \) is the set of all strings that can be obtained from strings in \( L \) by deleting exactly one \( 1 \). For example, if \( L = \{ 101101, 00, \varepsilon \} \), then \( \text{delete1}(L) = \{ 01101, 10101, 10110 \} \).

4. Consider the following recursively defined function on strings:

\[
\text{stutter}(w) := \begin{cases} 
\varepsilon & \text{if } w = \varepsilon \\
aa \cdot \text{stutter}(x) & \text{if } w = ax \text{ for some symbol } a \text{ and some string } x
\end{cases}
\]

Intuitively, \( \text{stutter}(w) \) doubles every symbol in \( w \). For example:

- \( \text{stutter(PRESTO)} = PPRREESSTTOO \)
- \( \text{stutter(HOCUS\diamondPOCUS)} = HHOOCCUUSS\diamondPPOOCCUUSS \)

Let \( L \) be an arbitrary regular language.

1. Prove that the language \( \text{stutter}^{-1}(L) := \{ w \mid \text{stutter}(w) \in L \} \) is regular.
2. Prove that the language \( \text{stutter}(L) := \{ \text{stutter}(w) \mid w \in L \} \) is regular.

5. Consider the following recursively defined function on strings:

\[
\text{evens}(w) := \begin{cases} 
\varepsilon & \text{if } w = \varepsilon \\
\varepsilon & \text{if } w = a \text{ for some symbol } a \\
b \cdot \text{evens}(x) & \text{if } w = abx \text{ for some symbols } a \text{ and } b \text{ and some string } x
\end{cases}
\]

Intuitively, \( \text{evens}(w) \) skips over every other symbol in \( w \). For example:

- \( \text{evens(EXPELLIARMUS)} = XELAMS \)
- \( \text{evens(AVADA\diamondKEDAVRA)} = VD\diamondEAR \).

Once again, let \( L \) be an arbitrary regular language.

1. Prove that the language \( \text{evens}^{-1}(L) := \{ w \mid \text{evens}(w) \in L \} \) is regular.
2. Prove that the language \( \text{evens}(L) := \{ \text{evens}(w) \mid w \in L \} \) is regular.