Show that NP is closed under the kleene-star operation.
CS/ECE-374: Lecture 28 - Final Exam review

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May 04, 2021

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Pre-lecture brain teaser

Show that NP is closed under the kleene-star operation.
Topics for the final exam include:

- Regular expressions
- DFAs, NFAs,
- Fooling Sets and Closure properties
- Turing Machines and Decidability
- Recursion and Dynamic Programming
- DFS/BFS
- Djikstra, Bellman-Ford (Path finding)
- Reductions/ NP-Completeness
In today’s lecture let’s focus on a few that you guys had trouble on in the midterms (and the most recent stuff which you’ll be tested on).

- Regular expressions
- DFAs, NFAs,
- **Fooling Sets and Closure properties**
- **Turing Machines and Decidability**
- Recursion and Dynamic Programming
- DFS/BFS
- Djikstra, Bellman-Ford (Path finding)
- **Reductions/ NP-Completeness**
Practice: Asymptotic bounds

Given an asymptotically tight bound for:

$$\sum_{i=1}^{n} i^3$$  \hspace{1cm} (1)
Find the regular expression for the language:

\[ \{ w \in \{0,1\}^* \mid \text{w does not contain 00 as a substring} \} \]
Is the following language regular?

\[ L = \{ w | w \text{ has an equal number of 0's and 1's } \} \]
Practice: NFAs and DFAs

Let $M$ be the following NFA:

Which of the following statements about $M$ are true?

1. $M$ accepts the empty string $\varepsilon$ - false
2. $\delta(s, 010) = \{s, a, c\}$ - true
3. $\varepsilon$-reach($a$) = $\{s, a, c\}$ - true
4. $M$ rejects the string $11100111000$ - true
5. $L(M) = (00)^* + (111)^*$ - true
Practice: NFAs and DFAs

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Practice: NFAs and DFAs

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4. M rejects the string $11100111000$ -
5. $L(M) = (00)^* + (111)^*$ -
Which of the following is true for every language $L \subseteq \{0, 1\}^*$

1. $L^*$ is non-empty -
2. $L^*$ is regular -
3. If $L$ is NP-Hard, then $L$ is not regular -
4. If $L$ is not regular, then $L$ is undecidable -
A *centipede* is an undirected graph formed by a path of length $k$ with two edges (legs) attached to each node on the path as shown in the below figure. Hence, the centipede graph has $3k$ vertices. The **CENTIPEDe** problem is the following: given an undirected graph $G = (V, E)$ and an integer $k$, does $G$ contain a *centipede* of $3k$ vertices as a subgraph? Prove that **CENTIPEDe** is NP-Complete.
What do we need to do to prove Centipede is NP-Complete?
Prove Centipede is in **NP:**
Prove Centipede is in **NP-hard**: 
Prove (via reduction) that the following language is undecidable.

\[ \text{AcceptOrBust} = \{\langle M \rangle | M \text{ does not reject any input} \} \]

Your reduction must involve the \textbf{SelfHalts} problem which is known to be undecidable:

\[ \text{SelfHalts} = \{\langle M \rangle | M \text{ halts on input } \langle M \rangle \} \]
Practice: Decidability

\[
\text{AcceptOrBust} = \{\langle M \rangle | \text{M does not reject any input}\}
\]

\[
\text{SelfHalts} = \{\langle M \rangle | \text{M halts on input} \; \langle M \rangle \}
\]
Consider the two problems:

**Problem: 3SAT**

- **Instance:** Given a CNF formula $\varphi$ with $n$ variables, and $k$ clauses
- **Question:** Is there a truth assignment to the variables such that $\varphi$ evaluates to true

**Problem: Clique**

- **Instance:** A graph $G$ and an integer $k$.
- **Question:** Does $G$ has a clique of size $\geq k$?

Reduce 3SAT to CLIQUE
Given a graph $G$, a set of vertices $V'$ is:

*clique*: every pair of vertices in $V'$ is connected by an edge of $G$. 
Reduction: 3SAT to Clique

Bust out the reduction diagram:

\[
\begin{align*}
\mathcal{R} & \xrightarrow{I_X} I_Y & \mathcal{A}_Y & \xrightarrow{YES} \text{NO} \\
\mathcal{A}_X
\end{align*}
\]
Some thoughts:

- Clique is a fully connected graph and very similar to the independent set problem
- We want to have a clique with all the satisfying literals
  - Can’t have literal and its negation in same clique
  - Only need one satisfying literal per clique
Hence the reduction creates a undirected graph $G$:

- Nodes in $G$ are organized in $k$ groups of nodes. Each triple corresponds to one clause.
- The edges of $G$ connect all but:
  - nodes in the same triple
  - nodes with contradictory labels ($x_1$ and $\overline{x_1}$)
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$$\varphi = (x_1 \lor x_2) \land (\overline{x_1} \lor \overline{x_2}) \land (\overline{x_1} \lor x_2)$$
Reduction: 3SAT to Clique

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3SAT to Independent Set Reduction

Very similar to 3SAT to independent set reduction:

\[
\neg x_1 \lor x_2 \lor \neg x_1 \land (x_1 \lor \neg x_2 \lor x_3) \land (\neg x_1 \lor x_2 \lor x_4)
\]

**Figure 1:** Graph for \( \varphi = (\neg x_1 \lor x_2 \lor x_3) \land (x_1 \lor \neg x_2 \lor x_3) \land (\neg x_1 \lor x_2 \lor x_4) \)