Finishing touches!

- Part I: models of computation (reg exps, DFA/NFA, CFGs, TMs)
- Part II: (efficient) algorithm design
- **Part III: intractability via reductions**
    - **Undecidablity: problems that have no algorithms**
    - **NP-Completeness: problems unlikely to have efficient algorithms unless $P = NP$**

CS/ECE-374: Lecture 22 - Reductions

Lecturer: Nickvash Kani
Chat moderator: Samir Khan

April 15, 2021

University of Illinois at Urbana-Champaign

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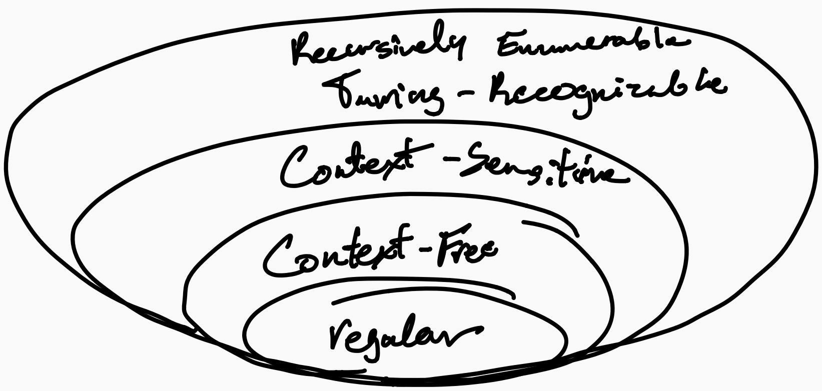
Turing defined TMs as a machine model of computation

Church-Turing thesis: any function that is computable can be computed by TMs

Efficient Church-Turing thesis: any function that is computable can be computed by TMs with only a polynomial slow-down



Computability and Complexity Theory

- What functions can and *cannot* be computed by TMs?
- What functions/problems can and cannot be solved *efficiently*?

Why?

- Foundational questions about computation
- Pragmatic: Can we solve our problem or not?
- Are we not being clever enough to find an efficient algorithm or should we stop because there isn't one or likely to be one?

Reductions to Prove Intractability

A general methodology to prove impossibility results.

- Start with some *known* hard problem X
- *Reduce* X to your favorite problem Y

If Y can be solved then so can $X \Rightarrow Y$ is also *hard*

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Caveat: In algorithms we reduce new problem to known solved one!

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If Y can be solved then so can $X \Rightarrow Y$ is also *hard*

Caveat: In algorithms we reduce new problem to known solved one!

Who gives us the initial hard problem?

- Some clever person (Cantor/Gödel/Turing/Cook/Levin ...) who establish hardness of a fundamental problem
- Assume some core problem is hard because we haven't been able to solve it for a long time. This leads to *conditional* results

Reduction Question

Given hard problem A   Want to prove A is hard

A general methodology to prove impossibility results.

- Start with some *known* hard problem X
- *Reduce* X to your favorite problem Y

$X \leq_p Y$

HALT $\leq$ A

If Y can be solved then so can $X \Rightarrow Y$ is also *hard*

What if we want to prove a problem is easy?

Find a easy Problem: E

$A \leq E$

Decision Problems, Languages, Terminology

When proving hardness we limit attention to *decision* problems

- A decision problem Π is a collection of instances (strings)
- For each instance I of Π, answer is YES or NO
- Equivalently: boolean function $f_\Pi : \Sigma^* \to \{0, 1\}$ where $f(I) = 1$ if I is a YES instance, $f(I) = 0$ if NO instance
- Equivalently: language $L_\Pi = \{I \mid I \text{ is a YES instance}\}$

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Notation about encoding: distinguish I from encoding $\langle I \rangle$

- n is an integer. $\langle n \rangle$ is the encoding of n in some format (could be unary, binary, decimal etc)
- G is a graph. $\langle G \rangle$ is the encoding of G in some format
- M is a TM. $\langle M \rangle$ is the encoding of TM as a string according to some fixed convention

Decision Problems, Languages, Terminology

Aside: Different problems can be formulated differently.

Example: Traveling Salesman

Common Formulation: Given a list of cities and the distances between each pair of cities, what is the shortest possible route that visits each city exactly once and returns to the origin city? C1 → C2 → C3 → ...

Decision Formulation: Given a list of cities and the distances between each pair of cities, is there a route route that visits each city exactly once and returns to the origin city **while having a shorter length than integer k.** Yes / No

Examples

- Given directed graph G, is it strongly connected? $\langle G \rangle$ is a YES instance if it is, otherwise NO instance
- Given number n, is it a prime number? $L_{PRIMES} = \{\langle n \rangle \mid n \text{ is prime}\}$
- Given number n is it a composite number? $L_{COMPOSITE} = \{\langle n \rangle \mid n \text{ is a composite}\}$
- Given $G = (V, E), s, t, B$ is the shortest path distance from s to t at most B? Instance is $\langle G, s, t, B \rangle$

Reductions: Overview

Reductions for decision problems|languages

For languages L_X, L_Y, a *reduction from* L_X *to* L_Y is:

- An algorithm ...
- Input: $w \in \Sigma^*$
- Output: $w' \in \Sigma^*$
- Such that:

$$\boxed{w \in L_X} \iff \boxed{w' \in L_Y}$$

Reductions for decision problems/languages

For decision problems X, Y, a *reduction from X to Y* is:

- An algorithm ...
- Input: I_X, an instance of X.
- Output: I_Y an instance of Y.
- Such that:
  $$\boxed{I_Y \text{ is YES instance of } Y} \iff \boxed{I_X \text{ is YES instance of } X}$$

Using reductions to solve problems

- $\mathcal{R}$: Reduction $X \to Y$
- $\mathcal{A}_Y$: algorithm for Y:

Using reductions to solve problems

- $\mathcal{R}$: Reduction $X \to Y$
- $\mathcal{A}_Y$: algorithm for Y:
- $\implies$ New algorithm for X:

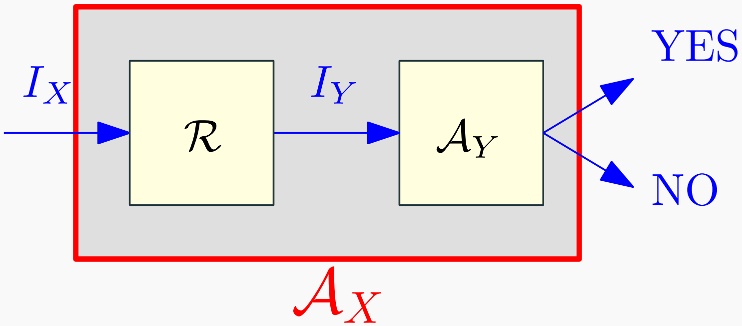
$\mathcal{A}_X(I_X)$:
    // I_X: instance of X.
    $I_Y \Leftarrow \mathcal{R}(I_X)$
    return $\mathcal{A}_Y(I_Y)$

Using reductions to solve problems

- $\mathcal{R}$: Reduction $X \rightarrow Y$
- $\mathcal{A}_Y$: algorithm for Y:
- $\implies$ New algorithm for X:

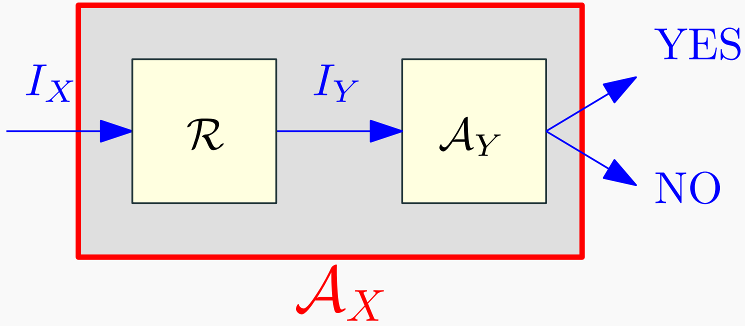
$X \leq Y$

```
𝒜_X(I_X):
        // I_X: instance of X.
        I_Y ⇐ ℛ(I_X)
        return 𝒜_Y(I_Y)
```



In particular, if $\mathcal{R}$ and $\mathcal{A}_Y$ are polynomial-time algorithms, $\mathcal{A}_X$ is also polynomial-time.

Reductions and running time



$R(n)$: running time of $\mathcal{R}$ $(n)^2$

$Q(n)$: running time of $\mathcal{A}_Y$

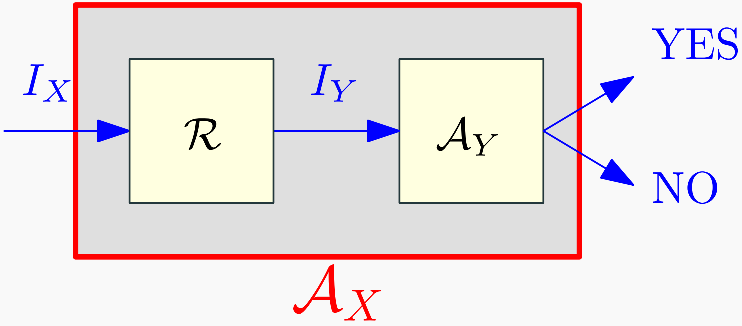
Question: What is running time of $\mathcal{A}_X$?

$|I_x| \equiv |I_y|$ $O(A_n) = O(R(n) + Q(n))$

$|I_x| = n$ $(|I_y| = O((n^2)$ $|I_y| = O(R(n))$

$A_y = (Q(R(n)))$

Reductions and running time



$R(n)$: running time of $\mathcal{R}$

$Q(n)$: running time of $\mathcal{A}_Y$

Question: What is running time of $\mathcal{A}_X$? $O(Q(R(n))$. Why?

- If I_X has size n, $\mathcal{R}$ creates an instance I_Y of size at most $R(n)$
- $\mathcal{A}_{\mathcal{Y}}$'s time on I_Y is by definition at most $Q(|I_Y|) \leq Q(R(n))$.

$A_Y = (n^2)^{1.5} = n^3$

Example: If $R(n) = n^2$ and $Q(n) = n^{1.5}$ then $\mathcal{A}_X$ is $O(n^2 + n^3)$

R A_Y

Comparing Problems

$X \leq_p Y$

- Reductions allow us to formalize the notion of "Problem X is no harder to solve than Problem Y".
- If Problem X reduces to Problem Y (we write $X \leq Y$), then X cannot be harder to solve than Y.
- More generally, if $X \leq Y$, we can say that X is no harder than Y, or Y is at least as hard as X. $X \leq Y$:
  - X is no harder than Y, or
  - Y is at least as hard as X.

Examples of Reductions

Independent Sets and Cliques

Given a graph G, a set of vertices V' is:

Independent Sets and Cliques

Given a graph G, a set of vertices V' is:

$G(V, E)$
$V' \subseteq V$

- An *independent set*: ifno two vertices of V' areconnected by an edgeof G.

Independent Sets and Cliques

Given a graph G, a set of vertices V' is:

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- *clique*: *every* pair of vertices in V' is connected by an edge of G.

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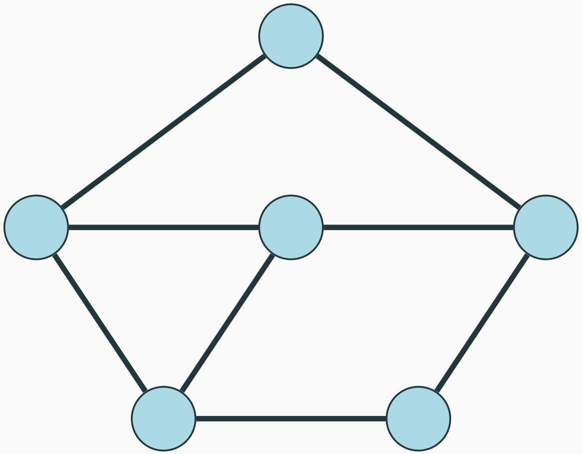

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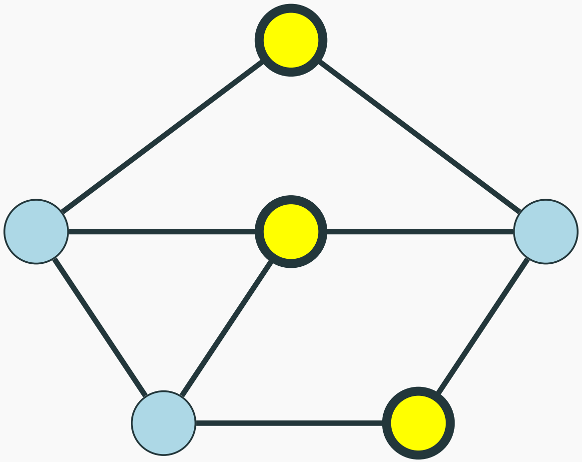

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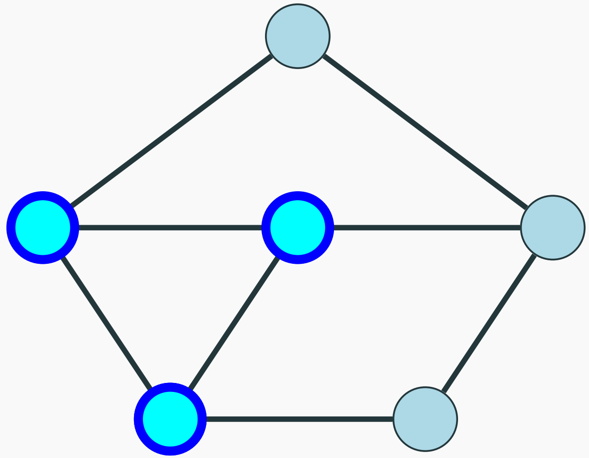


The Independent Set and Clique Problems

Problem: Independent Set

Instance: A graph G and an integer k.

Question: Does G has an independent set of size $\geq k$? Yes/No

The Independent Set and Clique Problems

Problem: Independent Set

> **Instance:** A graph G and an integer k.
> **Question:** Does G has an independent set of size $\geq k$?

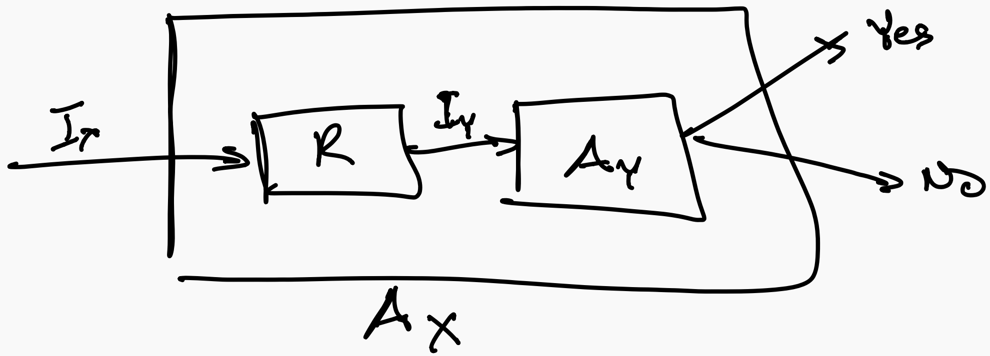
Problem: Clique

> **Instance:** A graph G and an integer k.
> **Question:** Does G has a clique of size $\geq k$?

Recall

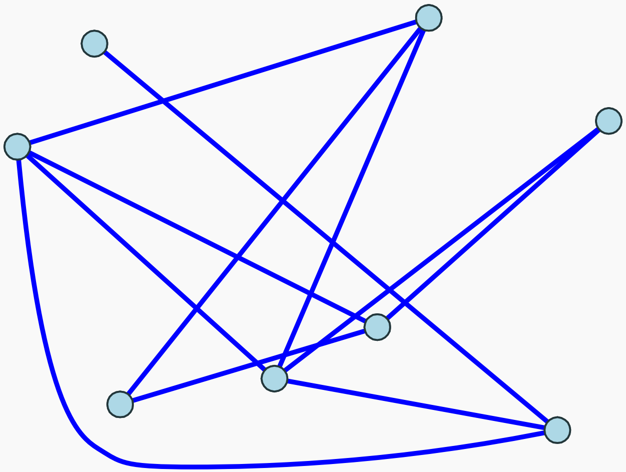
For decision problems X, Y, a reduction from X to Y is:

- An algorithm ...
- that takes I_X, an instance of X as input ...
- and returns I_Y, an instance of Y as output ...
- such that the solution (YES/NO) to I_Y is the same as the solution to I_X.



Reducing Independent Set to Clique

An instance of **Independent Set** is a graph G and an integer k.



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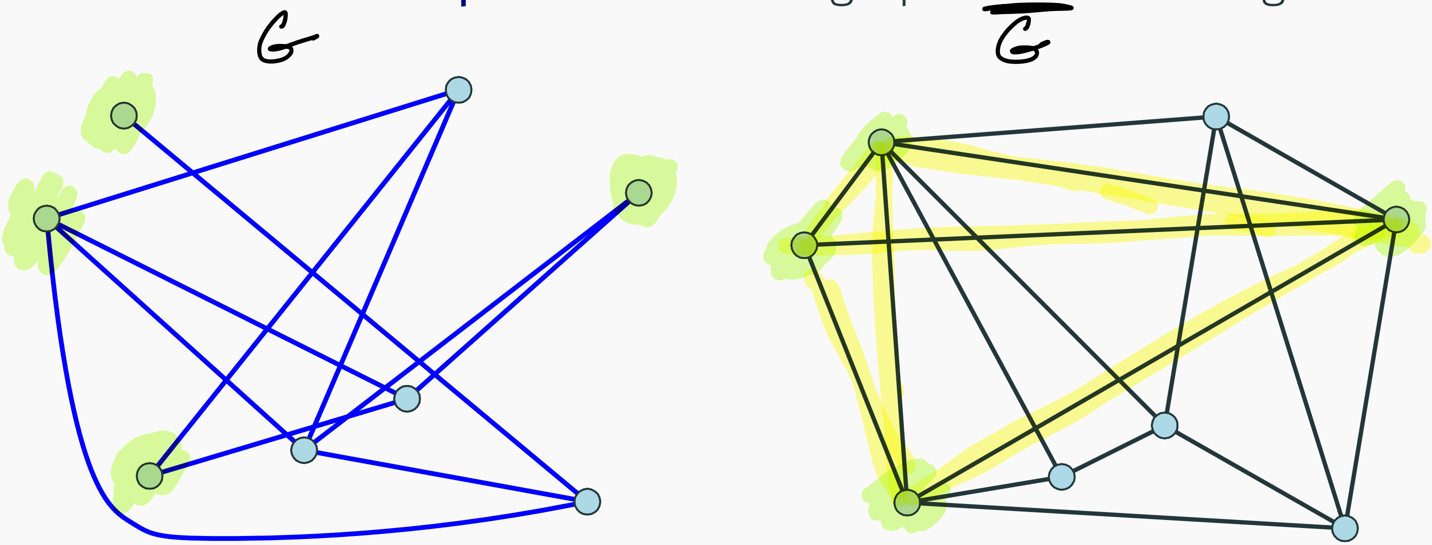
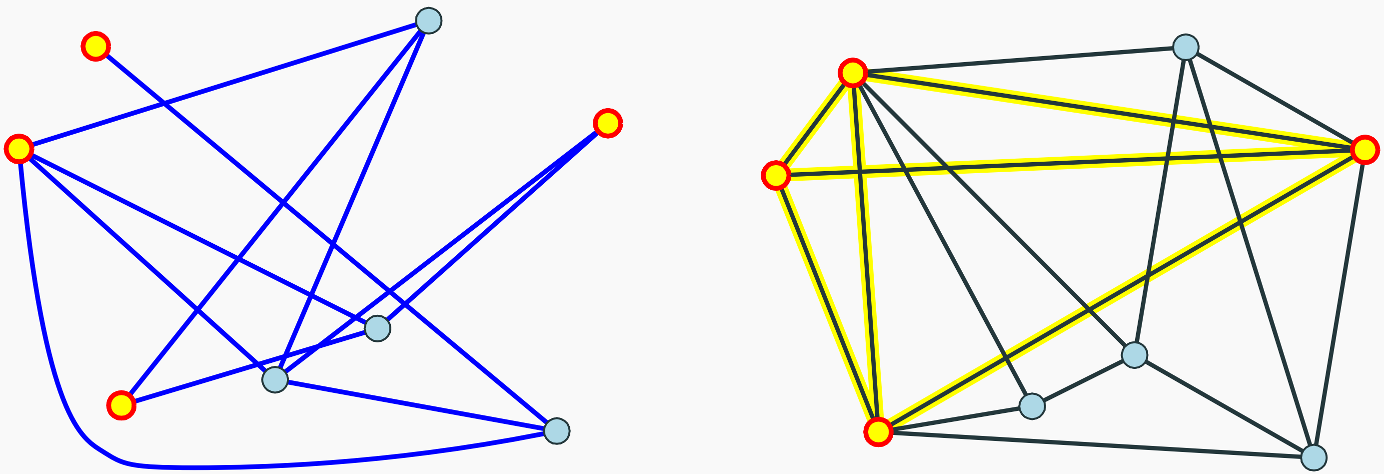


Reducing Independent Set to Clique

An instance of **Independent Set** is a graph G and an integer k.

Reduction given $\langle G, k \rangle$ outputs $\langle \overline{G}, k \rangle$ where $\overline{G}$ is the *complement* of G. $\overline{G}$ has an edge $uv \iff uv$ is not an edge of G.



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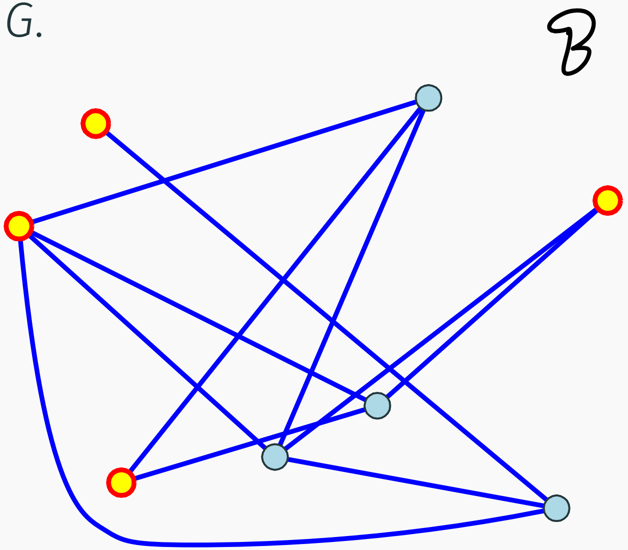

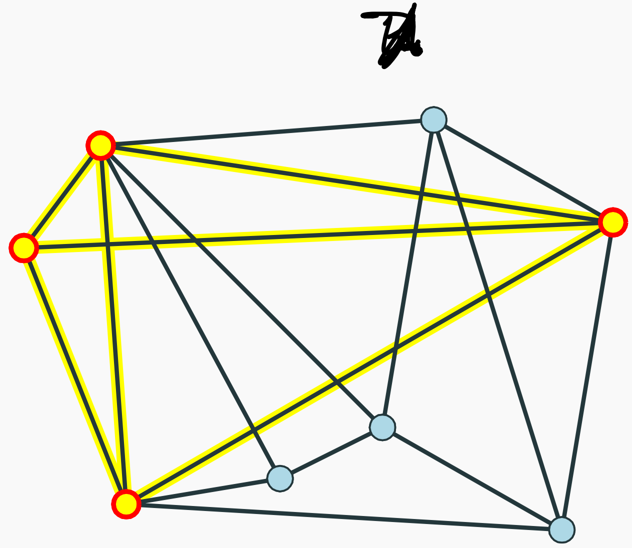

A independent set of size k in $G \iff$ A clique of size k in $\overline{G}$

Correctness of reduction

Lemma
G has an independent set of size $k \iff \overline{G}$ has a clique of size k.

Proof.
Need to prove two facts:

G has independent set of size at least k implies that $\overline{G}$ has a clique of size at least k.

$\overline{G}$ has a clique of size at least k implies that G has an independent set of size at least k.

Since $S \subseteq V$ is an independent set in $G \iff S$ is a clique in $\overline{G}$. □

Independent Set and Clique

- Independent Set $\leq_P$ Clique.

Independent Set and Clique

- **Independent Set** $\leq_P$ **Clique**.
  What does this mean?
- If have an algorithm for **Clique**, then we have an algorithm for **Independent Set**.

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  What does this mean?
- If have an algorithm for **Clique**, then we have an algorithm for **Independent Set**.
- **Clique** is *at least as hard as* **Independent Set**.

Independent Set and Clique

- **Independent Set** $\leq_P$ **Clique**.
  What does this mean?
- If have an algorithm for **Clique**, then we have an algorithm for **Independent Set**.
- **Clique** is *at least as hard as* **Independent Set**.
- Also... **Clique** $\leq_P$ **Independent Set**. Why? Thus **Clique** and **Independent Set** are polnomial-time equivalent.

Visualize Clique and independent Set Reduction

I want to show **Independent Set** is atleast as has as **Clique**.

Visualize Clique and independent Set Reduction

I want to show **Independent Set** is atleast as has as **Clique**.

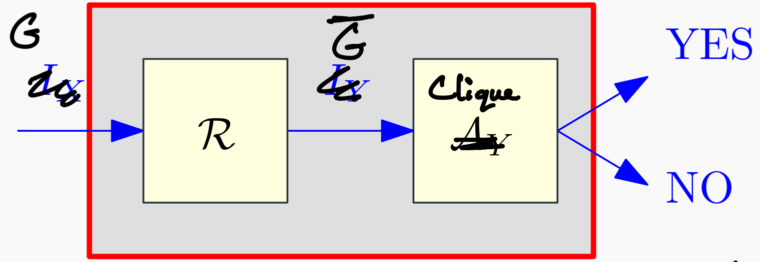
Write out the equality: **Clique** $\leq$ **Independent Set**

I want to show **Independent Set** is atleast as ~~has~~ hard as **Clique**.

Write out the equality: **Clique** $\leq$ **Independent Set**

A_x A_y

Draw reduction figure:



~~A_X~~ Independent Set

If Clique has poly solution then I.S. has polynomial solution

I.S. $\leq$ Clique

$I_x = \bar{G}$

A_x = Clique

$I_y = G$

A_Y = Independent set Problem

R: Transform $\bar{G}$ into G by taking the edge complement

Visualize Clique and independent Set Reduction

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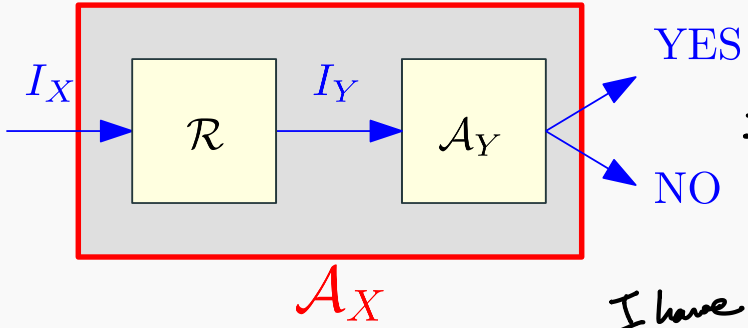
Write out the equality: **Clique** $\leq$ **Independent Set**

Draw reduction figure:



$\equiv \; X \leq Y$

I have Problem A
we know hardness of A

If want to prove that B is harder than A
$A \leq B$
$A_x = A$
$A_y = B$

If want to prove that A is harder than B?
(B is easier than A)
$B \leq A$
$A_x = B$
$A_y = A$

Fill in the blanks:

- $I_X = \langle \overline{G} \rangle$
- $A_X = \text{Clique}$
- $I_Y = \langle G \rangle$
- $A_Y = \text{Independent Set}$
- $R : \overline{G} = \{V, \overline{E}\}$

Review: Independent Set and Clique

Assume you can solve the **Clique** problem in $T(n)$ time. Then you can solve the **Independent Set** problem in

(A) $O(T(n))$ time.

(B) $O(n \log n + T(n))$ time.

(C) $O(n^2 T(n^2))$ time.

(D) $O(n^4 T(n^4))$ time.

(E) $O(n^2 + T(n^2))$ time.

(F) Does not matter - all these are polynomial if $T(n)$ is polynomial, which is good enough for our purposes.

Independent Set and Vertex Cover

Vertex Cover

Given a graph $G = (V, E)$, a set of vertices S is:

Vertex Cover

Given a graph $G = (V, E)$, a set of vertices S is:

- A *vertex cover* if every $e \in E$ has at least one endpoint in S.

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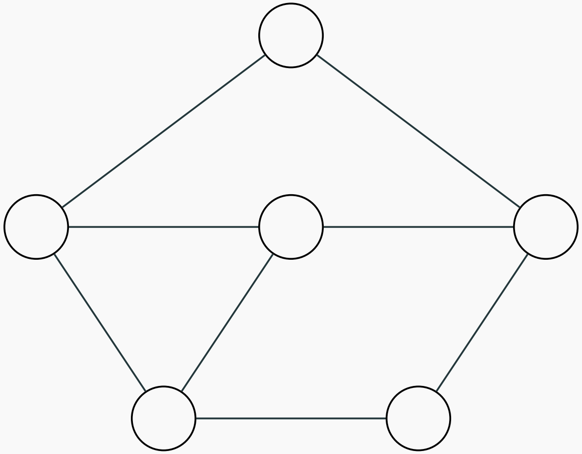

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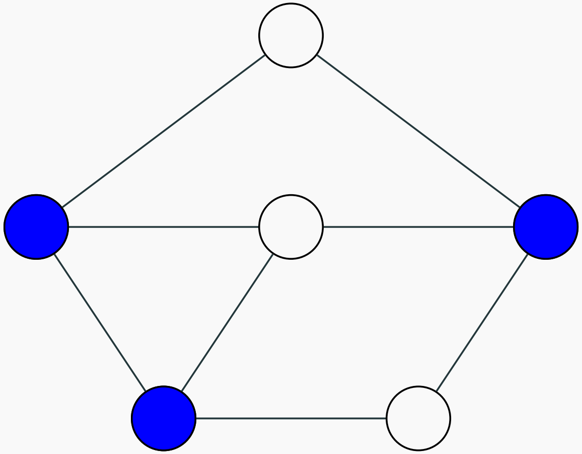

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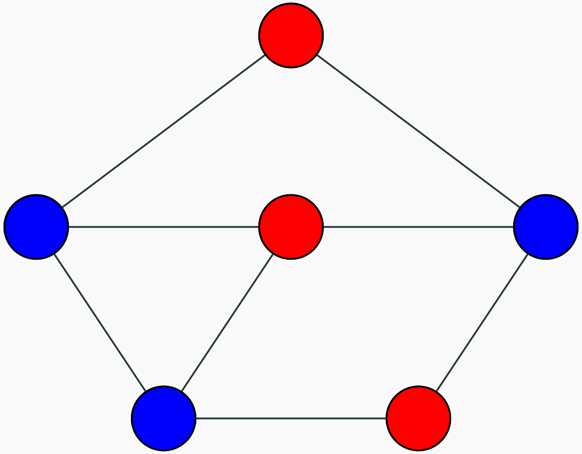

The Vertex Cover Problem

Problem (Vertex Cover)

Input: *A graph G and integer k.*

Goal: *Is there a vertex cover of size $\leq k$ in G?*

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Can we relate **Independent Set** and **Vertex Cover**?

Relationship between Vertex Cover and Independent Set

Lemma
Let $G = (V, E)$ be a graph. S is an Independent Set $\iff$ $V \setminus S$ is a vertex cover.

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Let $G = (V, E)$ be a graph. S is an Independent Set $\iff$ $V \setminus S$ is a vertex cover.

Proof.

$(\Rightarrow)$ Let S be an independent set

- Consider any edge $uv \in E$.
- Since S is an independent set, either $u \notin S$ or $v \notin S$.
- Thus, either $u \in V \setminus S$ or $v \in V \setminus S$.
- $V \setminus S$ is a vertex cover.

Relationship between Vertex Cover and Independent Set

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Let $G = (V, E)$ be a graph. S is an Independent Set $\iff$ $V \setminus S$ is a vertex cover.

Proof.

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- Since S is an independent set, either $u \notin S$ or $v \notin S$.
- Thus, either $u \in V \setminus S$ or $v \in V \setminus S$.
- $V \setminus S$ is a vertex cover.

($\Leftarrow$) Let $V \setminus S$ be some vertex cover:

- Consider $u, v \in S$
- uv is not an edge of G, as otherwise $V \setminus S$ does not cover uv.
- $\implies$ S is thus an independent set. □

Independent Set $\leq_P$ Vertex Cover

- G: graph with n vertices, and an integer k be an instance of the **Independent Set** problem.

Independent Set $\leq_P$ Vertex Cover

- G: graph with n vertices, and an integer k be an instance of the **Independent Set** problem.
- G has an independent set of size $\geq k \iff G$ has a vertex cover of size $\leq n - k$

Independent Set $\leq_P$ Vertex Cover

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- G has an independent set of size $\geq k \iff G$ has a vertex cover of size $\leq n - k$
- (G, k) is an instance of **Independent Set**, and $(G, n - k)$ is an instance of **Vertex Cover** with the same answer.

Independent Set $\leq_P$ Vertex Cover

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- G has an independent set of size $\geq k \iff G$ has a vertex cover of size $\leq n - k$
- (G, k) is an instance of **Independent Set**, and $(G, n - k)$ is an instance of **Vertex Cover** with the same answer.
- Therefore, **Independent Set** $\leq_P$ **Vertex Cover**. Also **Vertex Cover** $\leq_P$ **Independent Set**.

Independent Set $\leq_P$ Vertex Cover

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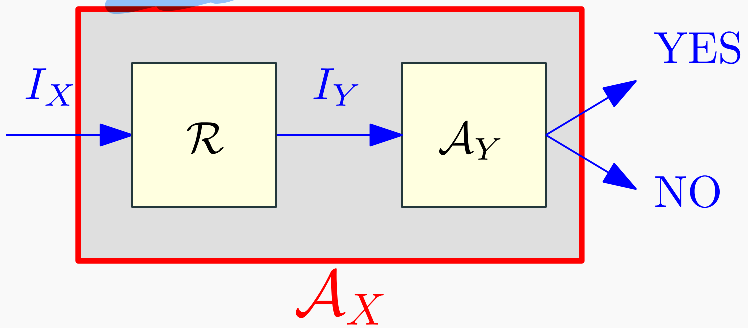


I.S ≤ V.C

V.C ≤ I.S

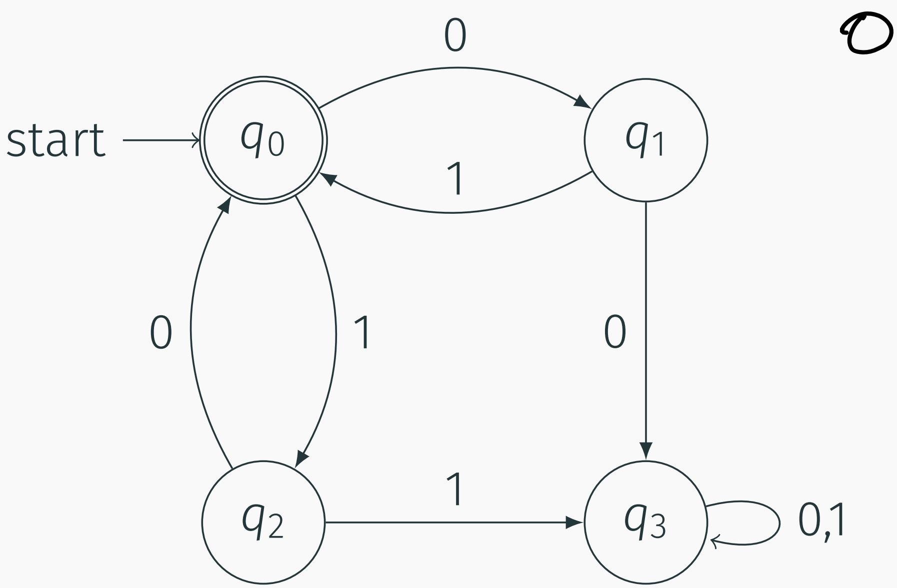
- $I_X = \langle G \rangle$
- $A_X = \text{Independent Set}(G, k)$
- $I_Y = \langle G \rangle$
- $A_Y = \text{Vertex Cover}(G, n - k)$
- $R : G' = G$

NFAs|DFAs and Universality

DFA Accepting a String

Given DFA M and string $w \in \Sigma^*$, does M accept w?

- Instance is $\langle M, w \rangle$
- Algorithm: given $\langle M, w \rangle$, output YES if M accepts w, else NO



Does above DFA accept 0010110? No

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Question: Is there an (efficient) algorithm for this problem?

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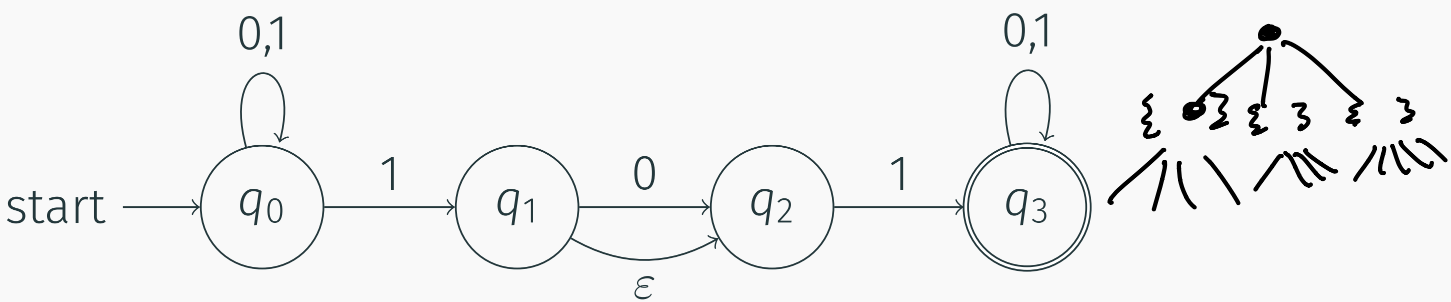
Question: Is there an (efficient) algorithm for this problem?

Yes. Simulate M on w and output YES if M reaches a final state.

Exercise: Show a linear time algorithm. Note that linear is in the input size which includes both encoding size of M and $|w|$.

Given NFA N and string $w \in \Sigma^*$, does N accept w?

- Instance is $\langle N, w \rangle$
- Algorithm: given $\langle N, w \rangle$, output YES if N accepts w, else NO



Does above NFA accept 0010110? Yes

NFA Accepting a String

Given NFA N and string $w \in \Sigma^*$, does N accept w?

- Instance is $\langle N, w \rangle$
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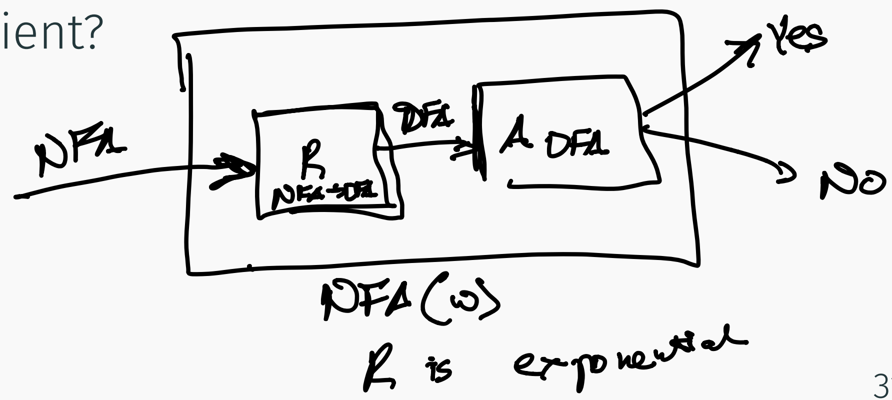
Question: Is there an algorithm for this problem? Brute Force

- Convert N to equivalent DFA M and use previous algorithm!
- Hence a reduction that takes $\langle N, w \rangle$ to $\langle M, w \rangle$
- Is this reduction efficient?

$A_{NFA(w)} \leq_e A_{DFA(w)}$

$O(2^n + n(2^n))$



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- Convert N to equivalent DFA M and use previous algorithm!
- Hence a reduction that takes $\langle N, w \rangle$ to $\langle M, w \rangle$
- Is this reduction efficient? No, because $|M|$ is exponential in $|N|$ in the worst case.

Exercise: Describe a polynomial-time algorithm.

Hence reduction may allow you to see an easy algorithm but not necessarily best algorithm!

DFA Universality

A DFA M is universal if it accepts every string.

That is, $L(M) = \Sigma^*$, the set of all strings.

Problem (DFA universality)

Input: *A DFA M.*

Goal: *Is M universal?*

How do we solve **DFA Universality**?

We check if M has *any* reachable non-final state.

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What is the problem with this reduction?

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Reduce it to **DFA Universality**?

Given an NFA N, convert it to an equivalent DFA M, and use the **DFA Universality** Algorithm.

What is the problem with this reduction? The reduction takes exponential time!

NFA Universality is known to be PSPACE-Complete.

Polynomial time reductions

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We say that an algorithm is *efficient* if it runs in polynomial-time.

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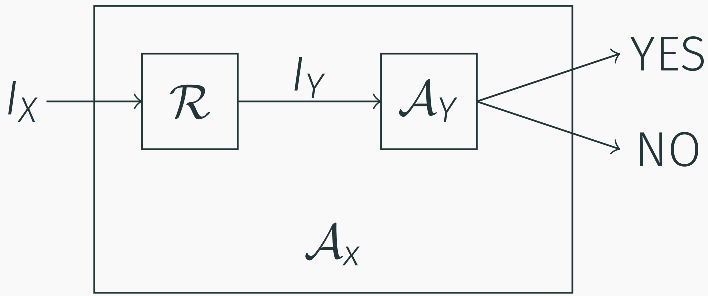

Polynomial-time Reduction

A polynomial time reduction from a *decision* problem X to a *decision* problem Y is an *algorithm* $\mathcal{A}$ that has the following properties:

- given an instance I_X of X, $\mathcal{A}$ produces an instance I_Y of Y
- $\mathcal{A}$ runs in time polynomial in $|I_X|$.
- Answer to I_X YES $\iff$ answer to I_Y is YES.

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Lemma
If $X \leq_P Y$ then a polynomial time algorithm for Y implies a polynomial time algorithm for X.

Such a reduction is called a *Karp reduction*. Most reductions we will need are Karp reductions.Karp reductions are the same as mapping reductions when specialized to polynomial time for the reduction step.

Review question: Reductions again...

Let X and Y be two decision problems, such that X can be solved in polynomial time, and $X \leq_P Y$. Then

(A) Y can be solved in polynomial time.

(B) Y can NOT be solved in polynomial time.

(C) If Y is hard then X is also hard.

(D) None of the above.

(E) All of the above.

Be careful about reduction direction

Note: $X \leq_P Y$ does not imply that $Y \leq_P X$ and hence it is very important to know the FROM and TO in a reduction.

To prove $X \leq_P Y$ you need to show a reduction FROM X TO Y

That is, show that an algorithm for Y implies an algorithm for X.

Turing machines and reductions

Reasoning about TMs/Programs

- $\langle M \rangle$ is encoding of a TM M.
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- Given $\langle M, w \rangle$ does M halt on input w?
- Given $\langle M, w \rangle$ does M accept w? (Universal Language)

Question: Do any of the above problems have an algorithm?

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Question: Do any of the above problems have an algorithm?

Theorem (Turing)
All the three problems are undecidable! No algorithm/program/TM.

Undecidability Reductions

CS 125 auto grading problem:

- student assignment: write program to print “Hello World”
- autograder: given student’s code $\langle S \rangle$ check if it prints “Hello World” correctly

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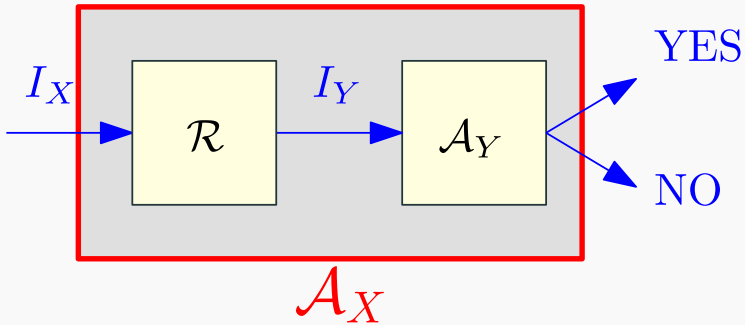
How do we reduce the halting problem to the autograding problem?! Want to prove $\text{HALT} \leq_P \text{Grader}$ ⟵ can't make for All programs



A_X = HALT

A_Y = Grader

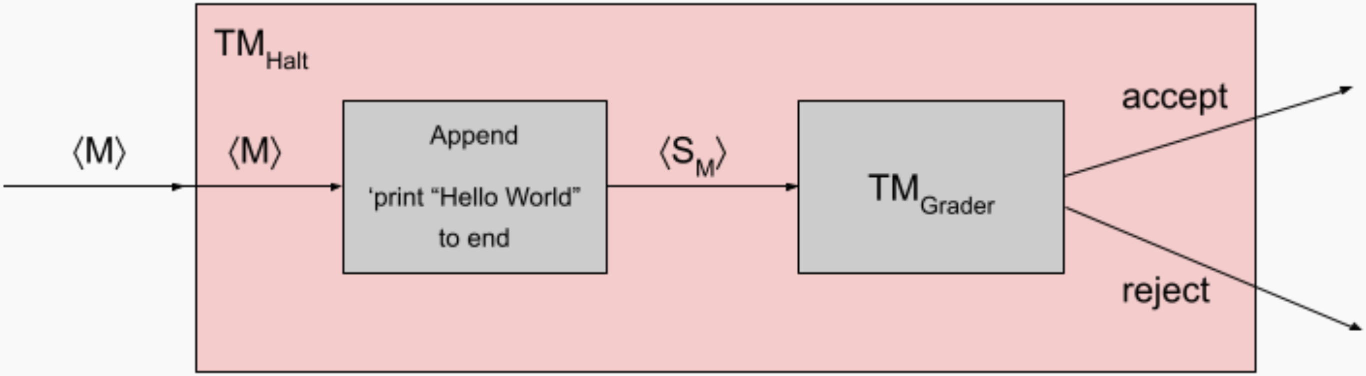
I_Y = <M + print ...>

I_X = <M>

R: Add "print "hello world"" at end of <M>

Undecidability Reductions

How do we reduce the halting problem to the autograding problem?! Want to prove $\mathrm{HALT} \leq_P \mathrm{Grader}$



Undecidability Reductions

CS 125 auto grading problem:

- student assignment: write program to print “Hello World”
- autograder: given student’s code $\langle S \rangle$ check if it prints “Hello World” correctly

Impossible! Why? Reduce Halting problem to CS125 autograding

Given arbitrary program $\langle M \rangle$ reduction generates program $\langle S_M \rangle$ such that S prints “Hello World” iff M halts

- Reduction is linear time algorithm. Just copies code of M to create code for S_M with additional couple of lines
- Main point: algorithm should work correctly for *every* input not just some simple cases.