Bellman-Ford and Dynamic Programming

Lecture 18
Part I

No negative edges: Dijkstra
Dijkstra’s Algorithm

Initialize for each node \( v \), \( \text{dist}(s, v) = \infty \)
Initialize \( X = \emptyset \), \( \text{dist}(s, s) = 0 \)
for \( i = 1 \) to \( |V| \) do
    Let \( v \) be such that \( \text{dist}(s, v) = \min_{u \in V - X} \text{dist}(s, u) \)
    \( X = X \cup \{v\} \)
    for each \( u \) in \( \text{Adj}(v) \) do
        \( \text{dist}(s, u) = \min(\text{dist}(s, u), \text{dist}(s, v) + \ell(v, u)) \)

Priority Queues to maintain \( \text{dist} \) values for faster running time

1. Using heaps and standard priority queues: \( O((m + n) \log n) \)
2. Best-first-search
Dijkstra’s Algorithm using Priority Queues

\[
\begin{align*}
Q & \leftarrow \text{makePQ}() \\
\text{insert}(Q, (s, 0)) & \text{for each node } u \neq s \text{ do} \\
\text{insert}(Q, (u, \infty)) & \text{X } \leftarrow \emptyset \\
\text{for } i = 1 \text{ to } |V| \text{ do} \\
(\nu, \text{dist}(s, \nu)) & = \text{extractMin}(Q) \\
X & = X \cup \{\nu\} \\
\text{for each } u \text{ in } \text{Adj}(\nu) \text{ do} \\
\text{decreaseKey}\left(Q, (u, \min(\text{dist}(s, u), \text{dist}(s, \nu) + \ell(\nu, u)))\right) & .
\end{align*}
\]

Priority Queue operations:

1. \(O(n)\) insert operations
2. \(O(n)\) extractMin operations
3. \(O(m)\) decreaseKey operations
Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

1. All operations can be done in $O(\log n)$ time
Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

- All operations can be done in \( O(\log n) \) time

Dijkstra’s algorithm can be implemented in \( O((n + m) \log n) \) time.
### Fibonacci Heaps

1. **extractMin, insert, delete, meld** in $O(\log n)$ time
2. **decreaseKey** in $O(1)$ **amortized** time:
Fibonacci Heaps

1. **extractMin**, **insert**, **delete**, **meld** in $O(\log n)$ time
2. **decreaseKey** in $O(1)$ amortized time: $\ell$ decreaseKey operations for $\ell \geq n$ take together $O(\ell)$ time
Fibonacci Heaps

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3. Relaxed Heaps: **decreaseKey** in $O(1)$ worst case time but at the expense of **meld** (not necessary for Dijkstra’s algorithm)
Fibonacci Heaps

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4. Dijkstra’s algorithm can be implemented in $O(n \log n + m)$ time.
Fibonacci Heaps

1. **extractMin**, **insert**, **delete**, **meld** in $O(\log n)$ time

2. **decreaseKey** in $O(1)$ amortized time: \( \ell \) decreaseKey operations for \( \ell \geq n \) take together $O(\ell)$ time

3. Relaxed Heaps: **decreaseKey** in $O(1)$ worst case time but at the expense of **meld** (not necessary for Dijkstra’s algorithm)

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1. Dijkstra’s algorithm can be implemented in $O(n \log n + m)$ time.

2. Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)
Key takeaways of Dijkstra

1. Non-negative edges: In order to get to $t$, only need nodes whose shortest distance is smaller than $t$. 

$\text{(UIUC)}$
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   - The intermediate set \( X \) keeps the \( i - 1 \) closest nodes.
Non-negative edges: In order to get to $t$, only need nodes whose shortest distance is smaller than $t$.

- The intermediate set $X$ keeps the $i - 1$ closest nodes.
- Give us an evaluation order: $d'(s, u)$ only updated when $v$ is added to $X$, and $u \in \text{Adj}(v)$ and $u \in V - X$. 

4
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   - In particular, once a node is in \( X \), \( d'(s, u) \) no longer changes as \( d'(s, u) = d(s, u) \), and it is never updated again.
Key takeaways of Dijkstra

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2. How to recognize the $i$-th closest node?
   \[ d'(s, u) = \min \left( d'(s, u), \text{dist}(s, v) + \ell(v, u) \right) \]
**Key takeaways of Dijkstra**

1. **Non-negative edges:** In order to get to \( t \), only need nodes whose shortest distance is smaller than \( t \).
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2. **How to recognize the \( i \)-th closest node?**
   \[ d'(s, u) = \min\left(d'(s, u), \text{dist}(s, v) + \ell(v, u)\right) \]
   - \( d'(s, u) \geq d(s, u) \)
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2. How to recognize the \( i \)-th closest node?
   \[ d'(s, u) = \min \left( d'(s, u), \ dist(s, v) + \ell(v, u) \right) \]
   - \( d'(s, u) \geq d(s, u) \)
   - \( d'(s, v) = \min_{u \in V - X} d'(s, u) \) is the \( i \)-th closest node, and \( d'(s, v) = d(s, v) \)
Part II

Negative Edges: Bellman-Ford
What are the distances computed by Dijkstra’s algorithm?

The distance as computed by Dijkstra algorithm starting from $s$:

- **A** $s = 0, x = 5, y = 1, z = 0$.
- **B** $s = 0, x = 1, y = 2, z = 5$.
- **C** $s = 0, x = 5, y = 1, z = 2$.
- **D** IDK.
Dijkstra’s Algorithm and Negative Lengths

With negative length edges, Dijkstra’s algorithm can fail.
Dijkstra’s Algorithm and Negative Lengths

With negative length edges, Dijkstra’s algorithm can fail

\[
X = \{ s, y \} \\
S \rightarrow y \rightarrow z \quad d'(s, z) = 2 < d'(s, x)
\]
Dijkstra’s Algorithm and Negative Lengths

With negative length edges, Dijkstra’s algorithm can fail

\[ X = \{ s, y, z \} \]
\[ x, w \]
\[ d'(s, w) = 3 \]
\[ < d'(s, x) \]
Dijkstra’s Algorithm and Negative Lengths

With negative length edges, Dijkstra’s algorithm can fail.
Dijkstra’s Algorithm and Negative Lengths

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Dijkstra’s Algorithm and Negative Lengths

With negative length edges, Dijkstra’s algorithm can fail.

False assumption: Dijkstra’s algorithm is based on the assumption that if $s = v_0 \rightarrow v_1 \rightarrow v_2 \ldots \rightarrow v_k$ is a shortest path from $s$ to $v_k$ then $\text{dist}(s, v_i) \leq \text{dist}(s, v_{i+1})$ for $0 \leq i < k$. Holds true only for non-negative edge lengths.

$$\text{d}(s, x) > \text{d}(s, z)$$
Anything we can learn from Dijkstra?

\[ d'(s, u) = \min\left( d'(s, u), \ \text{dist}(s, v) + \ell(v, u) \right) \]

- \( d'(s, u) \geq d(s, u) \) still true.
Anything we can learn from Dijkstra?

\[ d'(s, u) = \min(d'(s, u), \text{dist}(s, v) + \ell(v, u)) \]

- \( d'(s, u) \geq d(s, u) \) still true.

if \( s = v_0 \rightarrow v_1 \rightarrow v_2 \cdots \rightarrow v_k \) is a shortest path from \( s \) to \( v_k \)

- for \( 1 \leq i < k \): \( s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_i \) is a shortest path from \( s \) to \( v_i \), i.e. subpath of a shortest path is still a shortest path.

- Not true: \( \text{dist}(s, v_i) \leq \text{dist}(s, v_{i+1}) \), the intermediate set is no longer \( X \); in fact, it can be anything.
Anything we can learn from Dijkstra?

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- Not true: \( \text{dist}(s, v_i) \leq \text{dist}(s, v_{i+1}) \), the intermediate set is no longer \( X \); in fact, it can be anything

Solution: Update all edges \(|V| - 1\) times!
Bellman-Ford Algorithm

\[
\begin{align*}
\text{for each } u \in V & \text{ do} \\
& \quad d(u) \leftarrow \infty \\
& \quad d(s) \leftarrow 0 \\
\text{for } k = 1 \text{ to } n - 1 & \text{ do} \\
& \quad \text{for each } v \in V \text{ do} \\
& \quad \quad \text{for each edge } (u, v) \in \text{ln}(v) \text{ do} \\
& \quad \quad \quad d(v) = \min\{d(v), d(u) + \ell(u, v)\} \\
\text{for each } v \in V & \text{ do} \\
& \quad \text{dist}(s, v) \leftarrow d(v)
\end{align*}
\]

Running time: \( O(mn) \)
Part III

Bellman-Ford and DP
Shortest Paths and Recursion

1. Compute the shortest path distance from \( s \) to \( t \) recursively?
2. What are the smaller sub-problems?

Lemma
Let \( G \) be a directed graph with arbitrary edge lengths. If \( s = v_0 \to v_1 \to v_2 \to \ldots \to v_k \) is a shortest path from \( s \) to \( v_k \) then for \( 1 \leq i < k \):

\[
1. \quad s = v_0 \to v_1 \to v_2 \to \ldots \to v_i
\]

Sub-problem idea: paths of fewer hops/edges
Compute the shortest path distance from \( s \) to \( t \) recursively?

What are the smaller sub-problems?

**Lemma**

Let \( G \) be a directed graph with arbitrary edge lengths. If \( s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k \) is a shortest path from \( s \) to \( v_k \) then for \( 1 \leq i < k \):

\[ s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i \] is a shortest path from \( s \) to \( v_i \)
1. Compute the shortest path distance from $s$ to $t$ recursively?
2. What are the smaller sub-problems?

**Lemma**

Let $G$ be a directed graph with arbitrary edge lengths. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from $s$ to $v_k$, then for $1 \leq i < k$:

1. $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from $s$ to $v_i$

Sub-problem idea: paths of fewer hops/edges
Hop-based Recursion: Bellman-Ford Algorithm

Single-source problem: fix source $s$.

$d(v, k)$: shortest path length from $s$ to $v$ using at most $k$ edges.
Single-source problem: fix source $s$.

$d(v, k)$: shortest path length from $s$ to $v$ using at most $k$ edges.

Note: $\text{dist}(s, v) = d(v, n - 1)$. 

Hop-based Recursion: Bellman-Ford Algorithm
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Single-source problem: fix source $s$.
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Note: $\text{dist}(s, v) = d(v, n - 1)$.

Recursion for $d(v, k)$:
Hop-based Recursion: Bellman-Ford Algorithm

Single-source problem: fix source \( s \).
\( d(v, k) \): shortest path length from \( s \) to \( v \) using at most \( k \) edges.
Note: \( dist(s, v) = d(v, n - 1) \).

Recursion for \( d(v, k) \):

\[
d(v, k) = \min \begin{cases} 
\min_{u \in \text{In}(v)} (d(u, k - 1) + \ell(u, v)) . \\
\end{cases}
\]

Base case: \( d(s, 0) = 0 \) and \( d(v, 0) = \infty \) for all \( v \neq s \).
Example

\[
\begin{array}{cccccc}
& 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
S & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
a & 8 & 6 & 6 & 1 & -1 & -1 & -2 \\
b & 8 & 4 & 2 & 2 & 2 & 2 & 2 \\
c & 8 & 3 & 3 & 3 & 3 & 3 & 3 \\
d & 8 & 8 & 4 & 2 & 2 & 1 & 1 \\
e & 8 & 8 & 8 & 9 & 9 & 9 & 9 \\
f & 8 & 8 & 9 & 7 & 7 & 7 & 7 \\
d & \frac{11}{8} & \frac{11}{9} & 9 \\
e & \frac{3}{4} & \frac{3}{4} & 8 \\
f & \frac{8}{11} & \frac{8}{11} & 9 \\
d & \frac{8}{11} & \frac{8}{11} & 9 \\
e & \frac{8}{11} & \frac{8}{11} & 9 \\
f & \frac{8}{11} & \frac{8}{11} & 9 \\
\end{array}
\]

\[
\begin{array}{cccc}
S & b \rightarrow a & -3 \\
S & a \rightarrow b & 4 \\
S & c \rightarrow e & 3 \\
S & f \rightarrow c & -3 \\
\end{array}
\]
Bellman-Ford Algorithm

\[
\text{for each } u \in V \text{ do} \\
\quad d(u, 0) \leftarrow \infty \\
\quad d(s, 0) \leftarrow 0
\]

\[
\text{for } k = 1 \text{ to } n - 1 \text{ do} \\
\quad \text{for each } v \in V \text{ do} \\
\quad \quad d(v, k) \leftarrow d(v, k - 1) \\
\quad \quad \text{for each edge } (u, v) \in \text{In}(v) \text{ do} \\
\quad \quad \quad d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\}
\]

\[
\text{for each } v \in V \text{ do} \\
\quad \text{dist}(s, v) \leftarrow d(v, n - 1)
\]

Running time: \(O(mn)\)

Space: \(O(n^2)\)

Space can be reduced to \(O(n)\).
Bellman-Ford Algorithm

for each $u \in V$ do
  $d(u, 0) \leftarrow \infty$
  $d(s, 0) \leftarrow 0$

for $k = 1$ to $n - 1$ do
  for each $v \in V$ do
    $d(v, k) \leftarrow d(v, k - 1)$
    for each edge $(u, v) \in \text{In}(v)$ do
      $d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\}$

for each $v \in V$ do
  $\text{dist}(s, v) \leftarrow d(v, n - 1)$

Running time:

$O(mn)$

Space:

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Space can be reduced to $O(n)$. 
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\text{for each } u \in V \text{ do} \\
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Running time: \( O(mn) \)
Bellman-Ford Algorithm

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&\quad \quad d(v, k) \leftarrow d(v, k - 1) \\
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\end{align*}
\]

\[
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&\text{for each } v \in V \text{ do} \\
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Running time: $O(mn)$  
Space: $O(n^2)$
Bellman-Ford Algorithm

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&\qquad d(v, k) \leftarrow d(v, k - 1) \\
&\qquad \text{for each edge } (u, v) \in ln(v) \text{ do} \\
&\qquad\qquad d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\} \\
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\end{align*}
\]

Running time: \(O(mn)\)  Space: \(O(n^2)\)
Bellman-Ford Algorithm

```plaintext
for each \( u \in V \) do  
  \( d(u, 0) \leftarrow \infty \) 
  \( d(s, 0) \leftarrow 0 \)

for \( k = 1 \) to \( n - 1 \) do  
  for each \( v \in V \) do  
    \( d(v, k) \leftarrow d(v, k - 1) \)
  for each edge \( (u, v) \in \text{In}(v) \) do  
    \( d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\} \)

for each \( v \in V \) do  
  \( \text{dist}(s, v) \leftarrow d(v, n - 1) \)
```

Running time: \( O(mn) \)  
Space: \( O(n^2) \)

Space can be reduced to \( O(n) \).
Bellman-Ford Algorithm

\begin{algorithm}
\begin{algorithmic}
\State \textbf{for each} $u \in V$ \textbf{do}
\State \hspace{1em} $d(u) \leftarrow \infty$
\State \hspace{1em} $d(s) \leftarrow 0$
\State \textbf{for} $k = 1$ \textbf{to} $n - 1$ \textbf{do}
\State \hspace{1em} \textbf{for each} $v \in V$ \textbf{do}
\State \hspace{2em} \textbf{for each edge} $(u, v) \in \text{In}(v)$ \textbf{do}
\State \hspace{3em} $d(v) = \min\{d(v), d(u) + \ell(u, v)\}$
\State \textbf{for each} $v \in V$ \textbf{do}
\State \hspace{1em} $\text{dist}(s, v) \leftarrow d(v)$
\end{algorithmic}
\end{algorithm}

Running time: $O(mn)$ Space: $O(n)$
A cycle $C$ is a negative length cycle if the sum of the edge lengths of $C$ is negative.
Definition

A cycle $C$ is a negative length cycle if the sum of the edge lengths of $C$ is negative.
Shortest Paths and Negative Cycles

Given $G = (V, E)$ with edge lengths and $s, t$. Suppose

1. $G$ has a negative length cycle $C$, and
2. $s$ can reach $C$ and $C$ can reach $t$.

**Question:** What is the shortest distance from $s$ to $t$?
Shortest Paths and Negative Cycles

Given $G = (V, E)$ with edge lengths and $s, t$. Suppose

1. $G$ has a negative length cycle $C$, and
2. $s$ can reach $C$ and $C$ can reach $t$.

**Question:** What is the shortest distance from $s$ to $t$?

$-\infty$
Bellman-Ford: Negative Cycle Detection

Check if distances change in iteration $n$.

```
for each $u \in V$ do
    $d(u) \leftarrow \infty$
    $d(s) \leftarrow 0$

for $k = 1$ to $n - 1$ do
    for each $v \in V$ do
        for each edge $(u, v) \in ln(v)$ do
            $d(v) = \min\{d(v), d(u) + \ell(u, v)\}$

(* One more iteration to check if distances change *)

for each $v \in V$ do
    for each edge $(u, v) \in ln(v)$ do
        if $(d(v) > d(u) + \ell(u, v))$
            Output ‘‘Negative Cycle’’

for each $v \in V$ do
    $\text{dist}(s, v) \leftarrow d(v)$
```
Given directed graph $G$ with arbitrary edge lengths, does it have a negative length cycle?
Negative Cycle Detection

Given directed graph $G$ with arbitrary edge lengths, does it have a negative length cycle?

1. Bellman-Ford checks whether there is a negative cycle $C$ that is reachable from a specific vertex $s$. There may be negative cycles not reachable from $s$.
2. Run Bellman-Ford $|V|$ times, once from each node $u$?
Negative Cycle Detection

1. Add a new node $s'$ and connect it to all nodes of $G$ with zero length edges. Bellman-Ford from $s'$ will find a negative length cycle if there is one. **Exercise:** why does this work?

2. Negative cycle detection can be done with one Bellman-Ford invocation.