Independent Sets in Trees and Graph Basics

Lecture 15

How to design DP algorithms

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    1. Formulate the sub-problem
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2. Memoization
   1. Identify distinct subproblems
   2. Choose a memoization data structure
   3. Identify dependencies and find a good evaluation order
   4. An iterative algorithm replacing recursive calls with array lookups

Which data structure?

So far our memoization uses multi-dimensional arrays:

- Fibonacci numbers, 1-D array
- Text segmentation, suffix, 1-D array
- Longest increasing subsequence, suffix+index, 2-D array
- Edit distance, two prefixes, 2-D array

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Not always true.

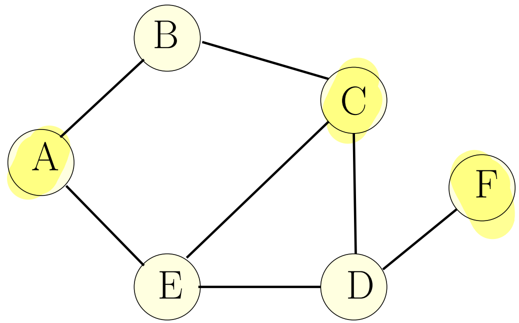
Part I

Maximum Weight Independent Set in Trees

Independent Set in a Graph

Definition

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is an independent set if there are no edges between nodes in S. That is, if $u, v \in S$ then $(u, v) \notin E$.

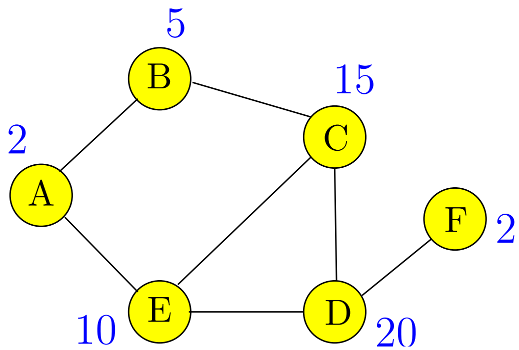


Some independent sets in graph above: $\{D\}, \{A, C\}, \{B, E, F\}$

Maximum Weight Independent Set Problem

Input Graph $G = (V, E)$ and weights $w(v) \geq 0$ for each $v \in V$

Goal Find maximum weight independent set in G



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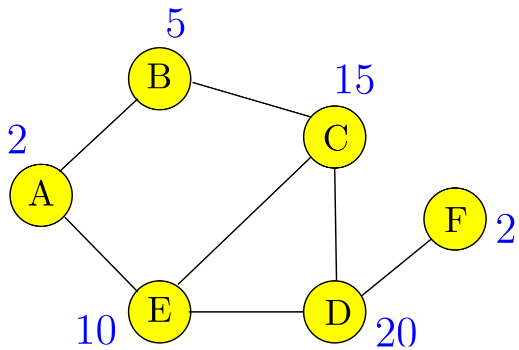

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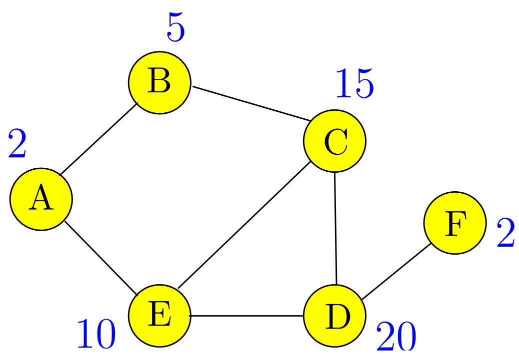

Some independent sets in graph above: $\{D\}, \{A, C\}, \{B, E, F\}$
Maximum weight independent set in above graph: $\{B, D\}$

25

Maximum Weight Independent Set Problem

- Finding the largest independent set in an arbitrary graph is extremely hard
- the canonical NP-hard problem

Backtracking

Convert into a sequence of decision problems.

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4. Find recursively optimum solutions without v_n (recurse on $G - v_n$) and with v_n (recurse on $G - v_n - N(v_n)$ & include v_n).

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5. If graph G is arbitrary there is no good ordering that resulted in a small number of subproblems.

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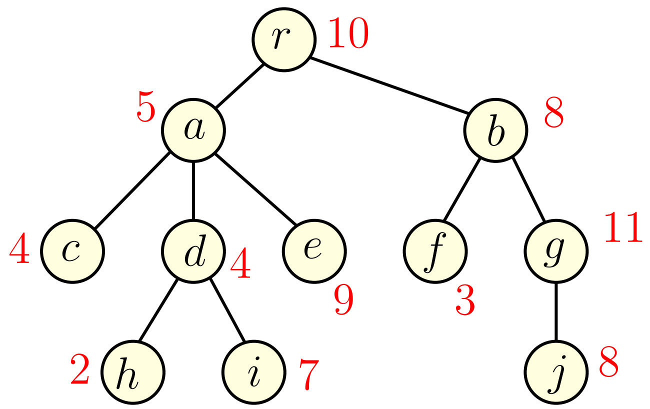
Maximum Weight Independent Set Problem

- Finding the largest independent set in an arbitrary graph is extremely hard
- the canonical NP-hard problem
- But in some special classes of graphs, we can find largest independent sets quickly
- when the input graph is a tree with n vertices, we can compute in O(n) time

Maximum Weight Independent Set in a Tree

Input Tree $T = (V, E)$ and weights $w(v) \geq 0$ for each $v \in V$

Goal Find maximum weight independent set in T



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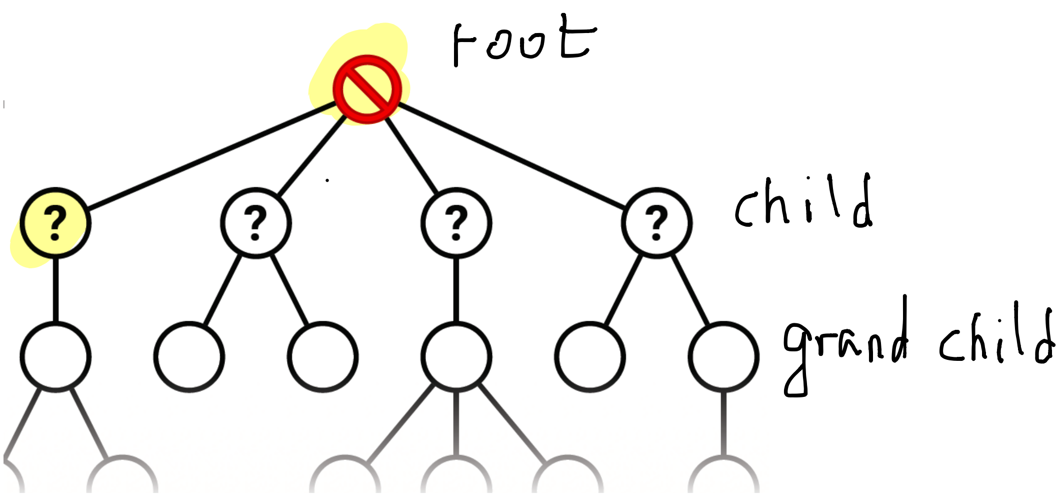
What is special about a tree?

Optimal substructure



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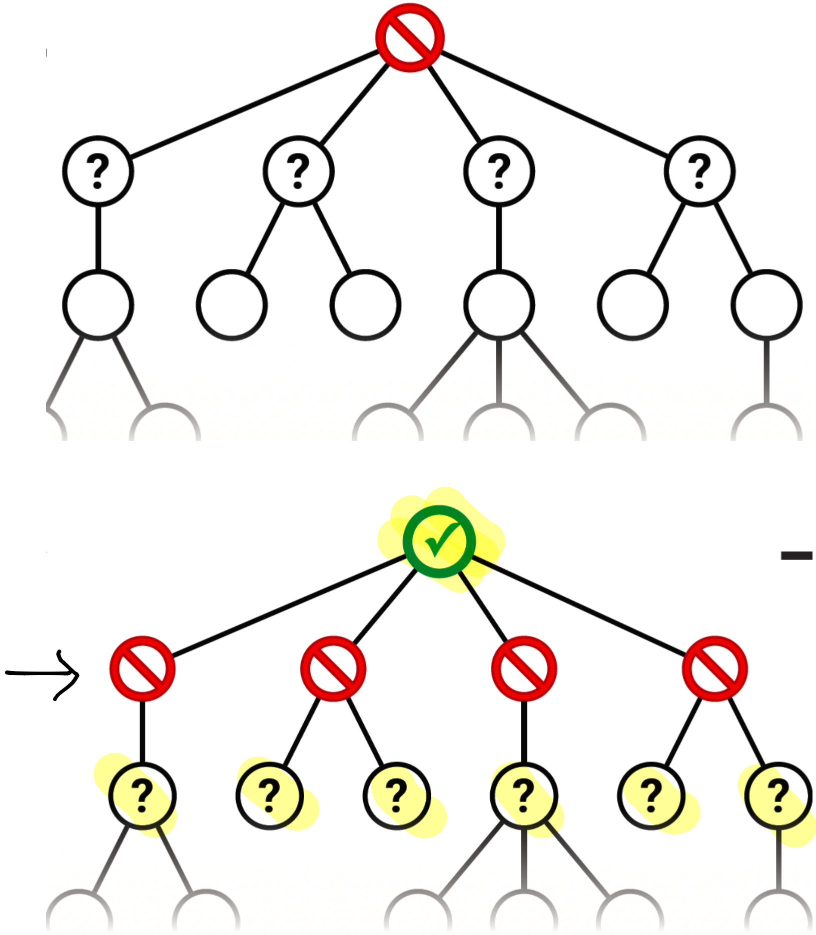

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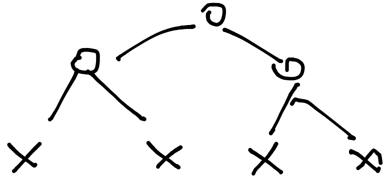
What data structure to memoize this recurrence? A tree



Order of evaluation

1. Compute $OPT(u)$ bottom up. To evaluate $OPT(u)$ need to have computed values of all children and grandchildren of u
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2. What is an ordering of nodes of a tree T to achieve above? Post-order traversal of a tree.

Iterative Algorithm

MIS-Tree(T):

    Let $v_1, v_2, \ldots, v_n$ be a post-order traversal of nodes of T

    for $i = 1$ to n **do**

$$M[v_i] = \max \left(\begin{array}{c} \sum_{v_j \text{ child of } v_i} M[v_j], \\ w(v_i) + \sum_{v_j \text{ grandchild of } v_i} M[v_j] \end{array} \right)$$

    return $M[v_n]$ (* Note: v_n is the root of T *)

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Space: $O(n)$ to store the value at each node of T

Running time:

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1. Naive bound: $O(n^2)$ since each $M[v_i]$ evaluation may take $O(n)$ time and there are n evaluations.

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Running time:

1. Naive bound: $O(n^2)$ since each $M[v_i]$ evaluation may take $O(n)$ time and there are n evaluations.
2. Better bound: $O(n)$. A value $M[v_j]$ is accessed only by its parent and grand parent.

Part II

Graph Basics

Why Graphs?

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1. Many important and useful optimization problems are graph problems
2. Two levels of resolution:
    1. Classic graph algorithms
    2. How to model a problem as a graph problem and solve it using the classic algorithms

Example: Medieval road network

Example: Modeling Problems as Search

State Space Search

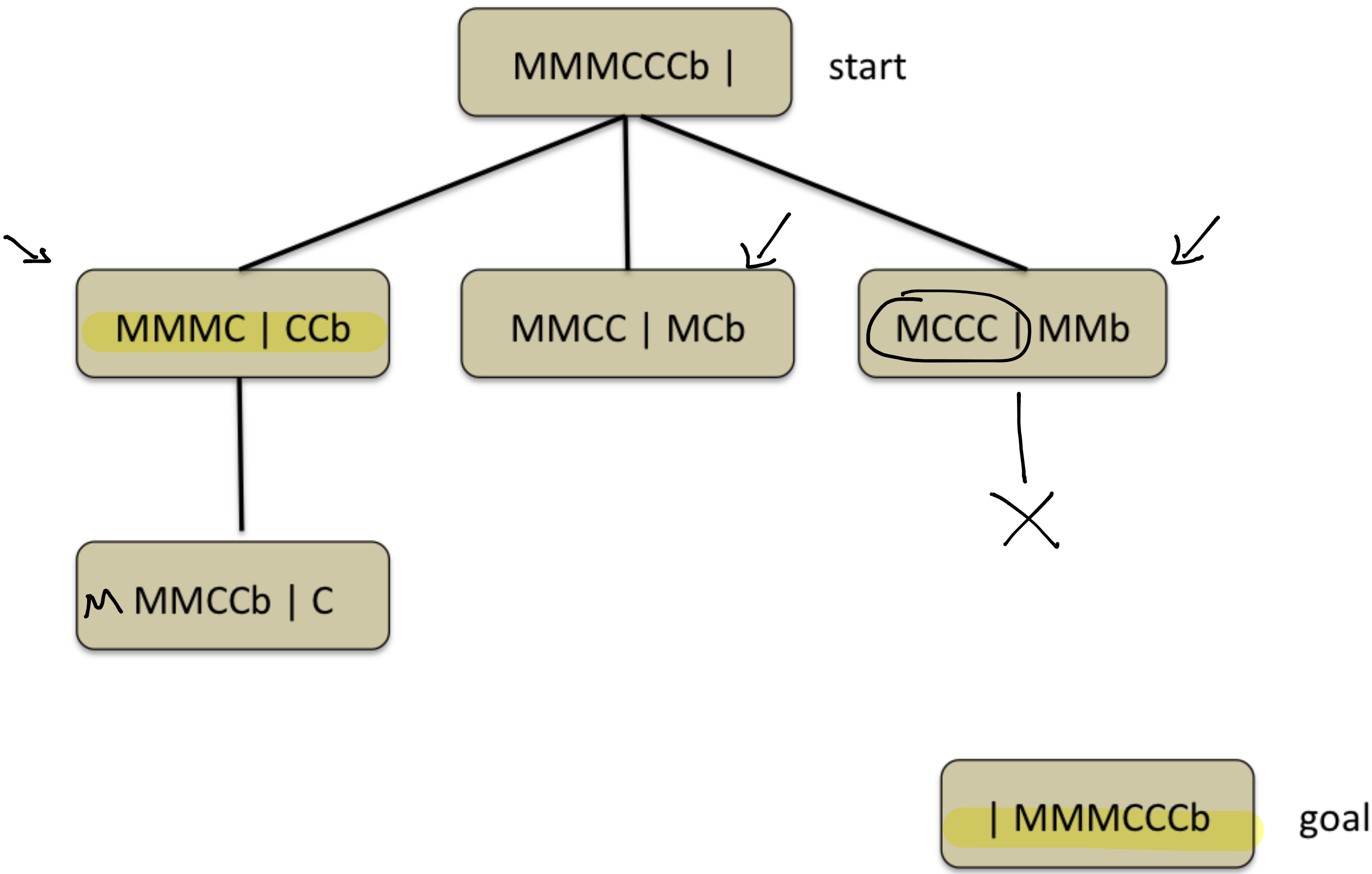
Many search problems can be modeled as search on a graph.
The trick is figuring out what the vertices and edges are.

Missionaries and Cannibals

- Three missionaries, three cannibals, one boat, one river
- Boat carries two people, must have at least one person
- Must all get across
- At no time can cannibals outnumber missionaries

How is this a graph search problem?
What are the vertices?
What are the edges?

Example: Missionaries and Cannibals Graph

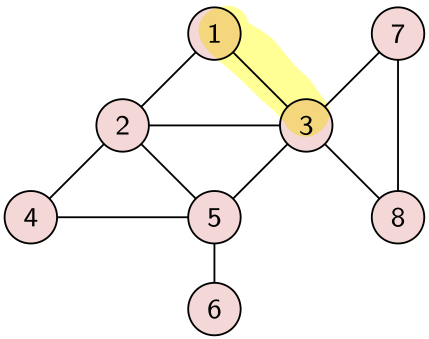


Graph

Definition

An undirected (simple) graph $G = (V, E)$ is a 2-tuple:

1. V is a set of vertices (also referred to as nodes)
2. E is a set of edges where each edge $e \in E$ is a set of the form $\{u, v\}$ with $u, v \in V$ and $u \neq v$.



Example

In figure, $G = (V, E)$ where $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 7\}, \{3, 8\}, \{4, 5\}, \{5, 6\}, \{7, 8\}\}$.

Notation and Convention

Graph is just a way of encoding pairwise relationships.

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Notation

An edge in an undirected graphs is an *unordered* pair of nodes and hence it is a set. Conventionally we use (u, v) for $\{u, v\}$ when it is clear from the context that the graph is undirected.

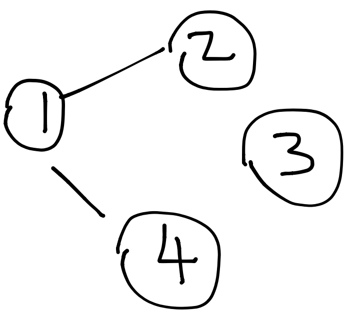
1. u and v are the end points of an edge $\{u, v\}$

Graph Representation I

Adjacency Matrix

Represent $G = (V, E)$ with n vertices and m edges using a $n \times n$ adjacency matrix A where

1. $A[i,j] = A[j,i] = 1$ if $\{i,j\} \in E$ and $A[i,j] = A[j,i] = 0$ if $\{i,j\} \notin E$.



$$\begin{bmatrix} 0 & 1 & & 1 \\ 1 & 0 & & \\ & & 0 & \\ 1 & & & 0 \end{bmatrix}$$

$\frac{n(n-1)}{2} = O(n^2)$ dense

$m = O(n)$ sparse

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2. Advantage: can check if $\{i, j\} \in E$ in $O(1)$ time
3. Disadvantage: needs $\Omega(n^2)$ space even when $m \ll n^2$

Graph Representation II

Adjacency Lists

Represent $G = (V, E)$ with n vertices and m edges using adjacency lists:

1. For each $u \in V$, $\text{Adj}(u) = \{v \mid \{u, v\} \in E\}$, that is neighbors of u. Sometimes $\text{Adj}(u)$ is the list of edges incident to u.

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2. Advantage: space is $O(m + n)$
3. Disadvantage: cannot "easily" determine in $O(1)$ time whether $\{i, j\} \in E$
   1. By sorting each list, one can achieve $O(\log n)$ time
   2. By hashing "appropriately", one can achieve $O(1)$ time

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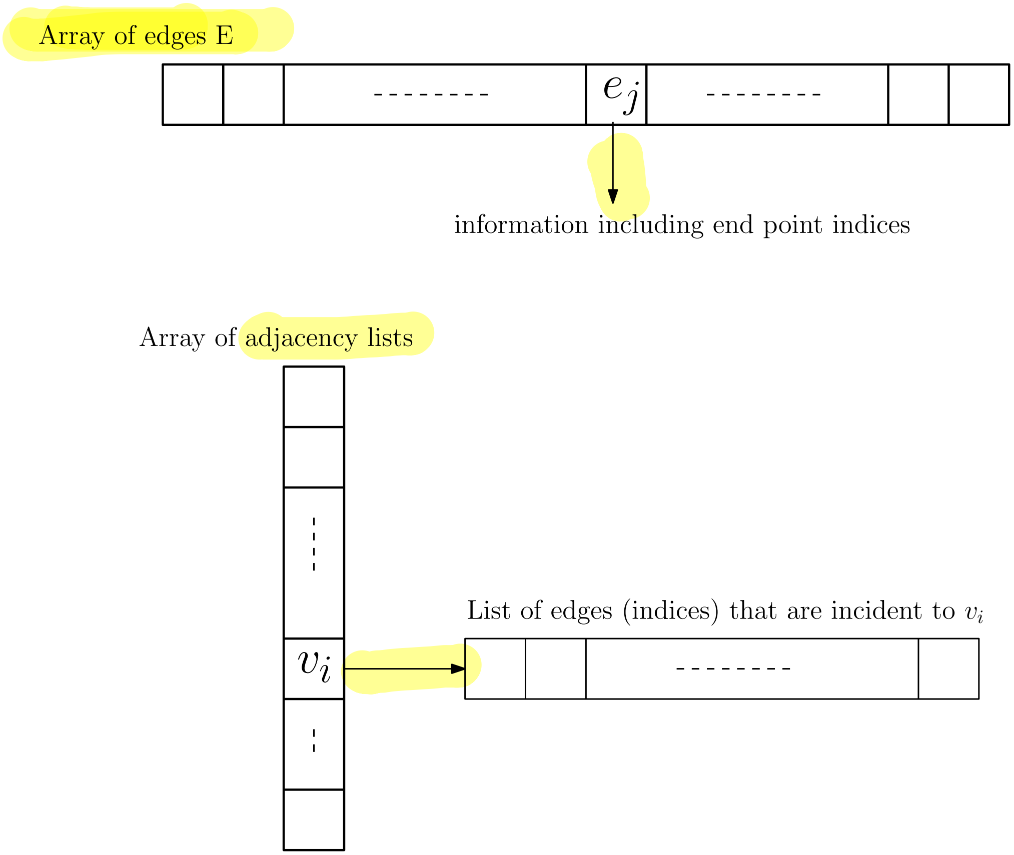
Note: In this class we will assume that by default, graphs are represented using plain vanilla (unsorted) adjacency lists.

A Concrete Representation

- Assume vertices are numbered arbitrarily as $\{1, 2, \ldots, n\}$.
- Edges are numbered arbitrarily as $\{1, 2, \ldots, m\}$.
- Edges stored in an array/list of size m. $E[j]$ is j'th edge with info on end points which are integers in range 1 to n.
- Array Adj of size n for adjacency lists. $Adj[i]$ points to adjacency list of vertex i. $Adj[i]$ is a list of edge indices in range 1 to m.

A Concrete Representation



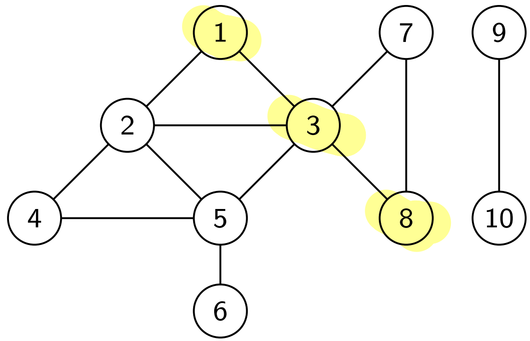
Connectivity Problems

Algorithmic Problems

1. Given graph G and nodes u and v, is u *connected* to v?
2. Given G and node u, find all nodes that are connected to u.
3. Find all connected components of G.

Connectivity on Undirected Graphs

Given a graph $G = (V, E)$:



1 3 5 2 3 8 ×

1 3 8 ✓

1 2 5 3 8 ✓

A path is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \le i \le k-1$. The length of the path is $k-1$ (the number of edges in the path) and the path is from v_1 to v_k.
Note: a single vertex u is a path of length 0.

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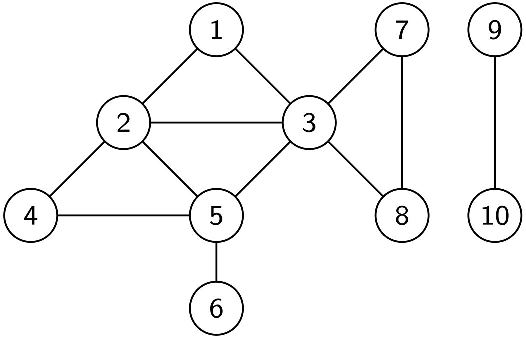

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A cycle is a sequence of *distinct* vertices $v_1, v_2, \ldots, v_k$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq k-1$ and $\{v_1, v_k\} \in E$. Single vertex not a cycle according to this definition.

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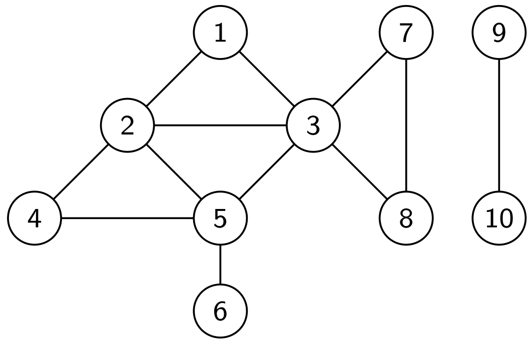

1 7

1 9

A vertex u is connected to v if there is a path from u to v.

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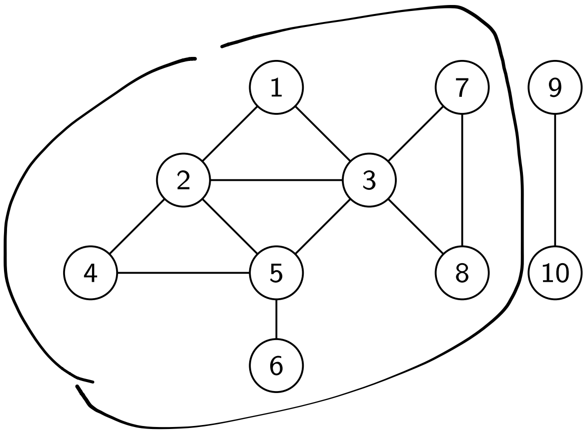

con(9)

con(1)

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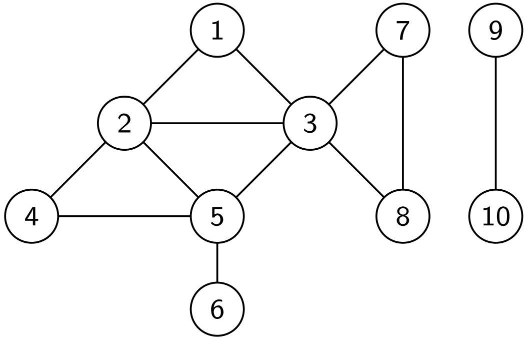
The connected component of u, $\mathrm{con}(u)$, is the set of all vertices connected to u.

Connectivity on Undirected Graphs

Define a relation C on $V \times V$ as uCv if u is connected to v

1. In undirected graphs, connectivity is a reflexive, symmetric, and transitive relation. Connected components are the equivalence classes.
2. Graph is connected if only one connected component.



Connectivity Problems on Undirected Graphs

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Can be accomplished in $O(m+n)$ time using **BFS** or **DFS**.
BFS and **DFS** are refinements of a basic search procedure which is good to understand on its own.