More DP: Text Segmentation and Edit Distance

Lecture 14
How to design DP algorithms

1. Find a “smart” recursion (The hard part)
   1. Formulate the sub-problem
   2. so that the number of distinct subproblems is small; polynomial in the original problem size.
How to design DP algorithms

1. Find a “smart” recursion (The hard part)
   1. Formulate the sub-problem
   2. so that the number of distinct subproblems is small; polynomial in the original problem size.

2. Memoization
   1. Identify distinct subproblems
   2. Choose a memoization data structure
   3. Identify dependencies and find a good evaluation order
   4. An iterative algorithm replacing recursive calls with array lookups
Part I

More Text Segmentation
A variation

Input A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function $\text{IsStringinL}(\text{string } x)$ that decides whether $x$ is in $L$, and non-negative integer $k$

Goal Decide if $w \in L^k$ using $\text{IsStringinL}(\text{string } x)$ as a black box sub-routine

Example

Suppose $L$ is $\text{English}$ and we have a procedure to check whether a string/word is in the $\text{English}$ dictionary.

- Is the string “isthisinanenglishsentence” in $\text{English}^5$?
- Is the string “isthisinanenglishsentence” in $\text{English}^4$?
- Is “asinineat” in $\text{English}^2$?
- Is “asinineat” in $\text{English}^4$?
- Is “zibzzzzad” in $\text{English}^1$?
Recursive Solution

When is \( w \in L^k \)?
Recursive Solution

When is $w \in L^k$?

$k = 0$: $w \in L^k$ iff $w = \epsilon$

$k = 1$: $w \in L^k$ iff $w \in L$

$k > 1$: $w \in L^k$ if $w = uv$ with $u \in L$ and $v \in L^{k-1}$
Recursive Solution

When is $w \in L^k$?

$k = 0$: $w \in L^k$ iff $w = \epsilon$

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$k > 1$: $w \in L^k$ if $w = uv$ with $u \in L$ and $v \in L^{k-1}$

Assume $w$ is stored in array $A[1..n]$

**IsStringinLk($A[1..n], k$):**

If $(k = 0)$

If $(n = 0)$ Output YES

Else Output NO

If $(k = 1)$

Output IsStringinL($A[1..n]$)

Else

For $(i = 1$ to $n - 1$) do

If (IsStringinL($A[1..i]$) and IsStringinLk($A[i + 1..n], k - 1)$)

Output YES

Output NO
IsStringinLk(A[1..n], k):
  If (k = 0)
    If (n = 0) Output YES
    Else Ouput NO
  If (k = 1)
    Output IsStringinL(A[1..n])
  Else
    For (i = 1 to n – 1) do
      If (IsStringinL(A[1..i]) and IsStringinLk(A[i + 1..n], k – 1))
        Output YES
    Output NO

- How many distinct sub-problems are generated by IsStringinLk(A[1..n], k)?
Analysis

**IsStringinLk(A[1..n], k):**

If $(k = 0)$
- If $(n = 0)$ Output YES
- Else Output NO

If $(k = 1)$
- Output **IsStringinL(A[1..n])**

Else
- For $(i = 1$ to $n - 1$) do
  - If (**IsStringinL(A[1..i])** and **IsStringinLk(A[i+1..n], k-1)**)
    - Output YES

Output NO

- How many distinct sub-problems are generated by **IsStringinLk(A[1..n], k)**? $O(nk)$
IsStringinLk(A[1..n], k):
    If (k = 0)
        If (n = 0) Output YES
        Else Output NO
    If (k = 1)
        Output IsStringinL(A[1..n])
    Else
        For (i = 1 to n - 1) do
            If (IsStringinL(A[1..i]) and IsStringinLk(A[i + 1..n], k - 1))
                Output YES
        Output NO

- How many distinct sub-problems are generated by IsStringinLk(A[1..n], k)? $O(nk)$
- How much space?
**Analysis**

\[
\text{IsStringinLk}(A[1..n], k) : \\
\text{If} (k = 0) \\
\quad \text{If} (n = 0) \text{ Output YES} \\
\quad \text{Else Output NO} \\
\text{If} (k = 1) \\
\quad \text{Output IsStringinL}(A[1..n]) \\
\text{Else} \\
\quad \text{For} (i = 1 \text{ to } n - 1) \text{ do} \\
\quad \quad \text{If} (\text{IsStringinL}(A[1..i]) \text{ and IsStringinLk}(A[i + 1..n], k - 1)) \\
\quad \quad \quad \text{Output YES} \\
\text{Output NO}
\]

- How many distinct sub-problems are generated by \text{IsStringinLk}(A[1..n], k)? \(O(nk)\)
- How much space? \(O(nk)\)
IsStringinLk\((A[1..n], k)\):
\[
\begin{align*}
\text{If} & \ (k = 0) \\
& \quad \text{If} \ (n = 0) \ \text{Output YES} \\
& \quad \text{Else Output NO} \\
\text{If} & \ (k = 1) \\
& \quad \text{Output IsStringinL}(A[1..n]) \\
\text{Else} \\
& \quad \text{For} \ (i = 1 \ \text{to} \ n - 1) \ \text{do} \\
& \quad \quad \text{If} \ (\text{IsStringinL}(A[1..i]) \ \text{and} \ \text{IsStringinLk}(A[i + 1..n], k - 1)) \\
& \quad \quad \quad \text{Output YES} \\
& \quad \text{Output NO}
\end{align*}
\]

- How many distinct sub-problems are generated by \(\text{IsStringinLk}(A[1..n], k)? O(nk)\)
- How much space? \(O(nk)\)
- Running time?
**IsStringinLk**$(A[1..n], k)$:

- If $(k = 0)$
  - If $(n = 0)$ Output YES
  - Else Output NO

- If $(k = 1)$
  - Output **IsStringinL**$(A[1..n])$

- Else
  - For $(i = 1$ to $n - 1$) do
    - If $(\text{IsStringinL}(A[1..i]) \text{ and } \text{IsStringinLk}(A[i + 1..n], k - 1))$
      - Output YES

Output NO

- How many distinct sub-problems are generated by **IsStringinLk**$(A[1..n], k)\? O(nk)$
- How much space? $O(nk)$
- Running time? $O(n^2k)$
Naming subproblems and recursive equation

\( ISL_k(i, h) \): a boolean which is 1 if \( A[i..n] \) is in \( L^h \), 0 otherwise

Base case: \( ISL_k(n+1, 0) = 1 \) interpreting \( A[n+1..n] \) as \( \epsilon \)
Naming subproblems and recursive equation

\( \text{ISL}_k(i, h) \): a boolean which is 1 if \( A[i..n] \) is in \( L^h \), 0 otherwise.

**Base case:** \( \text{ISL}_k(n + 1, 0) = 1 \) interpreting \( A[n + 1..n] \) as \( \epsilon \)

**Recursive relation:**
- \( \text{ISL}_k(i, h) = 1 \) if \( \exists i < j \leq n + 1 \) such that \( \text{ISL}_k(j, h - 1) = 1 \) and \( \text{IsStringinL}(A[i..(j - 1)]) = 1 \)
- \( \text{ISL}_k(i, h) = 0 \) otherwise

Alternately:
\[
\text{ISL}_k(i, h) = \max_{i < j \leq n+1} \text{ISL}_k(j, h - 1) \text{IsStringinL}(A[i..(j - 1)])
\]

**Output:** \( \text{ISL}_k(1, k) \)
How to order bottom up computation?

|   | 0 | 1 | 2 | 3 | h | K 
|---|---|---|---|---|---|---
| 1 |   |   |   |   |   | O
| 2 |   |   |   |   |   | O
| 3 |   |   |   |   |   | O
| 4 |   |   |   |   |   | O
| n |   |   |   |   |   | O
| n+1| 1 |   |   |   |   | O
Iterative Algorithm

\textbf{IsStringin$L^*$-Iterative}(A[1..n]):

boolean \ ISLk[1..(n + 1), 0...k] \\
\textbf{ISLk}[n + 1, 0] = TRUE \\
for (i = 1 to \ n) \\
\textbf{ISLk}[i, 0] = FALSE

for (h = 1 to \ k) \\
for (i = 1 to \ n) \\
\textbf{ISLk}[i, h] = FALSE \\
for (j = i + 1 to \ n + 1) \\
If (\textbf{ISLk}[j, h - 1] \text{ and } \textbf{IsStringinL}(A[i..j - 1])) \\
\textbf{ISLk}[i, h] = TRUE \\
Break

If (\textbf{ISLk}[1, k] = 1) Output YES \\
Else Output NO

\textbf{Running time: } O(n^2k). \textbf{ Space: } O(nk)
Another variant

Question: What if we want to check if $w \in L^i$ for some $0 \leq i \leq k$? That is, is $w \in \bigcup_{i=0}^{k} L^i$?
Part II

Edit Distance and Sequence Alignment
Spell Checking Problem

Given a string “exponen” that is not in the dictionary, how should a spell checker suggest a nearby string?
Spell Checking Problem

Given a string “exponen” that is not in the dictionary, how should a spell checker suggest a *nearby* string?

What does nearness mean?

**Question:** Given two strings $x_1 x_2 \ldots x_n$ and $y_1 y_2 \ldots y_m$ what is a *distance* between them?
Spell Checking Problem

Given a string “exponen” that is not in the dictionary, how should a spell checker suggest a nearby string?

What does nearness mean?

**Question:** Given two strings $x_1 x_2 \ldots x_n$ and $y_1 y_2 \ldots y_m$ what is a distance between them?

**Edit Distance:** minimum number of “edits” to transform $x$ into $y$. 
Edit distance between two words $X$ and $Y$ is the number of letter insertions, letter deletions and letter substitutions required to obtain $Y$ from $X$.

Example

The edit distance between FOOD and MONEY is at most 4:

$\text{FOOD} \rightarrow \text{MOOD} \rightarrow \text{MONOD} \rightarrow \text{MONED} \rightarrow \text{MONEY}$
Alignment

Place words one on top of the other, with gaps in the first word indicating insertions, and gaps in the second word indicating deletions.

\[
\begin{array}{cccc}
F & O & O & D \\
M & O & N & E \\
\end{array}
\]
Alignment

Place words one on top of the other, with gaps in the first word indicating insertions, and gaps in the second word indicating deletions.

\[
\begin{array}{cccc}
F & O & O & D \\
M & O & N & E & Y
\end{array}
\]

Formally, an alignment is a set \( M \) of pairs \((i, j)\) such that each index appears at most once, and there is no “crossing”: \( i < i' \) and \( i \) is matched to \( j \) implies \( i' \) is matched to \( j' > j \). In the above example, this is \( M = \{(1, 1), (2, 2), (3, 3), (4, 5)\} \).
Alignment

Place words one on top of the other, with gaps in the first word indicating insertions, and gaps in the second word indicating deletions.

F O O D
M O N E Y

Formally, an alignment is a set $M$ of pairs $(i, j)$ such that each index appears at most once, and there is no “crossing”: $i < i'$ and $i$ is matched to $j$ implies $i'$ is matched to $j' > j$. In the above example, this is $M = \{(1, 1), (2, 2), (3, 3), (4, 5)\}$.

Cost of an alignment is the number of columns that do not contain the same letter twice.
Edit Distance Problem

Problem
Given two words, find the edit distance between them, i.e., an alignment of smallest cost.
Applications

1. Spell-checkers and Dictionaries
2. Unix diff
3. DNA sequence alignment ... but, we need a new metric
## Similarity Metric

### Definition

For two strings $X$ and $Y$, the cost of alignment $M$ is

1. **[Gap penalty]** For each gap in the alignment, we incur a cost $\delta$.
2. **[Mismatch cost]** For each pair $p$ and $q$ that have been matched in $M$, we incur cost $\alpha_{pq}$; typically $\alpha_{pp} = 0$.

Edit distance is a special case when $\alpha_{pq} = 1$. 
Definition

For two strings $X$ and $Y$, the cost of alignment $M$ is

1. [Gap penalty] For each gap in the alignment, we incur a cost $\delta$.
2. [Mismatch cost] For each pair $p$ and $q$ that have been matched in $M$, we incur cost $\alpha_{pq}$; typically $\alpha_{pp} = 0$.

Edit distance is special case when $\delta = \alpha_{pq} = 1$. 
An Example

Example

\[ \text{Cost} = \delta + \alpha_{ae} \]

Alternative:

\[ \text{Cost} = 3\delta \]

Or a really stupid solution (delete string, insert other string):

\[ \text{Cost} = 19\delta. \]
What is the minimum edit distance for the following two strings, if insertion/deletion/change of a single character cost 1 unit?

What is the edit distance between...

374
473

1
2
3
4
5
What is the edit distance between...

What is the minimum edit distance for the following two strings, if insertion/deletion/change of a single character cost $1$ unit?

373
473

1
2
3
4
5
What is the edit distance between...

What is the minimum edit distance for the following two strings, if insertion/deletion/change of a single character cost 1 unit?

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

37 473
Sequence Alignment

Input  Given two words $X$ and $Y$, and gap penalty $\delta$ and mismatch costs $\alpha_{pq}$

Goal  Find alignment of minimum cost
Edit distance

Basic observation

Let $X = \alpha x$ and $Y = \beta y$

$\alpha, \beta$: strings.

$x$ and $y$ single characters.

Think about optimal edit distance between $X$ and $Y$ as alignment, and consider last column of alignment of the two strings:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$x$</th>
<th>or</th>
<th>$\alpha$</th>
<th>$x$</th>
<th>or</th>
<th>$\alpha x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>$y$</td>
<td></td>
<td>$\beta y$</td>
<td></td>
<td></td>
<td>$\beta$</td>
<td>$y$</td>
</tr>
</tbody>
</table>

Observation

*Prefixes must have optimal alignment!*
Try all possibilities

Observation

Let $X = x_1 x_2 \cdots x_m$ and $Y = y_1 y_2 \cdots y_n$. If $(m, n)$ are not matched then either the $m$th position of $X$ remains unmatched or the $n$th position of $Y$ remains unmatched.

1. Case $x_m$ and $y_n$ are matched.
   - Pay mismatch cost $\alpha_{x_m y_n}$ plus cost of aligning strings $x_1 \cdots x_{m-1}$ and $y_1 \cdots y_{n-1}$

2. Case $x_m$ is unmatched.
   - Pay gap penalty plus cost of aligning $x_1 \cdots x_{m-1}$ and $y_1 \cdots y_n$

3. Case $y_n$ is unmatched.
   - Pay gap penalty plus cost of aligning $x_1 \cdots x_m$ and $y_1 \cdots y_{n-1}$
Recursive Algorithm

Assume $X$ is stored in array $A[1..m]$ and $Y$ is stored in $B[1..n]$. Array $COST$ stores cost of matching two chars. Thus $COST[a, b]$ give the cost of matching character $a$ to character $b$.

\[
\begin{align*}
EDIST(A[1..m], B[1..n]) & = \\
\text{If } (m = 0) & \text{ return } n\delta \\
\text{If } (n = 0) & \text{ return } m\delta \\
m_1 & = \delta + EDIST(A[1..(m-1)], B[1..n]) \\
m_2 & = \delta + EDIST(A[1..m], B[1..(n-1)]) \\
m_3 & = COST[A[m], B[n]] + EDIST(A[1..(m-1)], B[1..(n-1)]) \\
\text{return } & \min(m_1, m_2, m_3)
\end{align*}
\]
Recursive Algorithm

\[
EDIST(A[1..m], B[1..n])
\]

1. If \( (m = 0) \) return \( n\delta \)
2. If \( (n = 0) \) return \( m\delta \)
3. \( m_1 = \delta + EDIST(A[1..(m-1)], B[1..n]) \)
4. \( m_2 = \delta + EDIST(A[1..m], B[1..(n-1)]) \)
5. \( m_3 = COST[A[m], B[n]] + EDIST(A[1..(m-1)], B[1..(n-1)]) \)
6. return \( \min(m_1, m_2, m_3) \)

How many distinct sub-problems will \( EDIST(A[1..m], B[1..n]) \) generate?
Recursive Algorithm

\[ \text{EDIST}(A[1..m], B[1..n]) \]
- If \( m = 0 \) return \( n\delta \)
- If \( n = 0 \) return \( m\delta \)

\[ m_1 = \delta + \text{EDIST}(A[1..(m-1)], B[1..n]) \]
\[ m_2 = \delta + \text{EDIST}(A[1..m], B[1..(n-1)]) \]
\[ m_3 = \text{COST}[A[m], B[n]] + \text{EDIST}(A[1..(m-1)], B[1..(n-1)]) \]

return \( \min(m_1, m_2, m_3) \)

How many distinct sub-problems will \( \text{EDIST}(A[1..m], B[1..n]) \) generate? \( O(nm) \)

\( \text{UIUC} \)
EDIST\((A[1..m], B[1..n])\)

If \(m = 0\) return \(n\delta\)
If \(n = 0\) return \(m\delta\)

\[m_1 = \delta + EDIST(A[1..(m - 1)], B[1..n])\]
\[m_2 = \delta + EDIST(A[1..m], B[1..(n - 1)])\]
\[m_3 = COST[A[m], B[n]] + EDIST(A[1..(m - 1)], B[1..(n - 1)])\]

return \(\min(m_1, m_2, m_3)\)

- How many distinct sub-problems will \(EDIST(A[1..m], B[1..n])\) generate? \(O(nm)\)
- What is the running time if we memoize recursion?
Recursive Algorithm

\[ \text{EDIST}(A[1..m], B[1..n]) \]
- If \( m = 0 \) return \( n\delta \)
- If \( n = 0 \) return \( m\delta \)
- \( m_1 = \delta + \text{EDIST}(A[1..(m - 1)], B[1..n]) \)
- \( m_2 = \delta + \text{EDIST}(A[1..m], B[1..(n - 1)]) \)
- \( m_3 = \text{COST}[A[m], B[n]] + \text{EDIST}(A[1..(m - 1)], B[1..(n - 1)]) \)
- return \( \min(m_1, m_2, m_3) \)

- How many distinct sub-problems will \( \text{EDIST}(A[1..m], B[1..n]) \) generate? \( O(nm) \)
- What is the running time if we memoize recursion? \( O(nm) \) since each call takes \( O(1) \) time to assemble the answers from recursive calls and no other computation.
Recursive Algorithm

**EDIST**(*A*[1..*m*], *B*[1..*n*])

- If (*m* = 0) return *n*δ
- If (*n* = 0) return *m*δ
- *m*₁ = δ + **EDIST**(A[1..(*m* – 1)], *B*[1..*n*])
- *m*₂ = δ + **EDIST**(A[1..*m*], *B*[1..(*n* – 1)])
- *m*₃ = **COST**[*A*[m], *B*[n]] + **EDIST**(A[1..(*m* – 1)], *B*[1..(*n* – 1)])
- return min(*m*₁, *m*₂, *m*₃)

- How many distinct sub-problems will **EDIST**(A[1..*m*], B[1..*n*]) generate? **O**(nm)
- What is the running time if we memoize recursion? **O**(nm)
since each call takes **O**(1) time to assemble the answers from to recursive calls and no other computation.
- How much space for memoization?
Recursive Algorithm

\[ EDIST(A[1..m], B[1..n]) \]

If \((m = 0)\) return \(n\delta\)
If \((n = 0)\) return \(m\delta\)
\[ m_1 = \delta + EDIST(A[1..(m - 1)], B[1..n]) \]
\[ m_2 = \delta + EDIST(A[1..m], B[1..(n - 1)]) \]
\[ m_3 = COST[A[m], B[n]] + EDIST(A[1..(m - 1)], B[1..(n - 1)]) \]
return \(\min(m_1, m_2, m_3)\)

- How many distinct sub-problems will \( EDIST(A[1..m], B[1..n]) \) generate? \( O(nm) \)
- What is the running time if we memoize recursion? \( O(nm) \) since each call takes \( O(1) \) time to assemble the answers from to recursive calls and no other computation.
- How much space for memoization? \( O(nm) \)
After seeing that number of subproblems is $O(nm)$ we name them to help us understand the structure better.
Naming subproblems and recursive equation

After seeing that number of subproblems is \( O(nm) \) we name them to help us understand the structure better.

**Optimal Costs**

Let \( \text{Opt}(i, j) \) be optimal cost of aligning \( x_1 \cdots x_i \) and \( y_1 \cdots y_j \). Then

\[
\text{Opt}(i, j) = \min \left\{ \alpha_{x_i, y_j} + \text{Opt}(i - 1, j - 1), \right. \\
\left. \delta + \text{Opt}(i - 1, j), \right. \\
\left. \delta + \text{Opt}(i, j - 1) \right\}
\]
After seeing that number of subproblems is $O(nm)$ we name them to help us understand the structure better.

**Optimal Costs**

Let $\text{Opt}(i, j)$ be optimal cost of aligning $x_1 \cdots x_i$ and $y_1 \cdots y_j$. Then

$$\text{Opt}(i, j) = \min \left\{ \begin{array}{l}
\alpha_{x_i, y_j} + \text{Opt}(i - 1, j - 1), \\
\delta + \text{Opt}(i - 1, j), \\
\delta + \text{Opt}(i, j - 1) \end{array} \right. $$

Base Cases: $\text{Opt}(i, 0) = \delta \cdot i$ and $\text{Opt}(0, j) = \delta \cdot j$
How to order bottom up computation?

Base case: \( \text{Opt}(i, 0) = \delta \cdot i \) and \( \text{Opt}(0, j) = \delta \cdot j \)

Recursive relation: Fill in row by row (or column by column)
Removing Recursion to obtain Iterative Algorithm

\[ int \ M[0..m][0..n] \]

Initialize all entries of \( M[i][j] \) to \( \infty \)

return \( EDIST(A[1..m], B[1..n]) \)

\[
EDIST(A[1..m], B[1..n])
\]

\[ int \ M[0..m][0..n] \]

for \( i = 1 \) to \( m \) do \( M[i, 0] = i \delta \)

for \( j = 1 \) to \( n \) do \( M[0, j] = j \delta \)

for \( i = 1 \) to \( m \) do

for \( j = 1 \) to \( n \) do

\[
M[i][j] = \min \left\{ \alpha_{x_i y_j} + M[i - 1][j - 1], \delta + M[i - 1][j], \delta + M[i][j - 1] \right\}
\]

Running time: \( O(nm) \)

Space: \( O(nm) \)
Removing Recursion to obtain Iterative Algorithm

```
int M[0..m][0..n]
Initialize all entries of M[i][j] to ∞
return EDIST(A[1..m], B[1..n])
```

```
EDIST(A[1..m], B[1..n])
int M[0..m][0..n]
for i = 1 to m do M[i, 0] = iδ
for j = 1 to n do M[0, j] = jδ

for i = 1 to m do
    for j = 1 to n do
        M[i][j] = min
            \(\alpha_{x_i,y_j} + M[i-1][j-1]\),
            \(\delta + M[i-1][j]\),
            \(\delta + M[i][j-1]\)
```

Running time: \(O(nm)\)
Space: \(O(nm)\)
Typically the DNA sequences that are aligned are about $10^5$ letters long!

So about $10^{10}$ operations and $10^{10}$ bytes needed

The killer is the 10GB storage

Can we reduce space requirements?
Recall

\[ M(i, j) = \min \left\{ \begin{array}{l}
\alpha_{x_i y_j} + M(i - 1, j - 1), \\
\delta + M(i - 1, j), \\
\delta + M(i, j - 1)
\end{array} \right. \]

Entries in \( j \)th column only depend on \((j - 1)\)st column and earlier entries in \( j \)th column

Only store the current column and the previous column reusing space; \( N(i, 0) \) stores \( M(i, j - 1) \) and \( N(i, 1) \) stores \( M(i, j) \)
Computing in column order to save space

Figure: $M(i, j)$ only depends on previous column values. Keep only two columns and compute in column order.
Space Efficient Algorithm

for all $i$ do $N[i, 0] = i\delta$

for $j = 1$ to $n$ do

$N[0, 1] = j\delta$ (* corresponds to $M(0, j)$ *)

for $i = 1$ to $m$ do

$N[i, 1] = \min \left\{ \alpha_{x_i y_j} + N[i - 1, 0], \delta + N[i - 1, 1], \delta + N[i, 0] \right\}$

for $i = 1$ to $m$ do

Copy $N[i, 0] = N[i, 1]$

Analysis

Running time is $O(mn)$ and space used is $O(2m) = O(m)$
Which data structure?

So far our memoization uses multi-dimensional arrays:

- Fibonacci numbers, 1-D array
- Text segmentation, suffix, 1-D array
- Longest increasing subsequence, suffix+index, 2-D array
- Edit distance, two prefixes, 2-D array
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Not always true.