Dynamic Programming

Lecture 13

Recursion types

1. **Divide and Conquer**: Problem reduced to multiple **independent** sub-problems.

   Examples: Merge sort, quick sort, multiplication, median selection.

   Each sub-problem is a fraction smaller.

2. **Backtracking**: A sequence of decision problems. Recursion tries all possibilities at each step.

   Each subproblem is only a constant smaller, e.g. from n to $n - 1$.

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3. **Dynamic Programming**: Smart recursion with memoization

Part I

Fibonacci Numbers

Fibonacci Numbers

Fibonacci numbers defined by recurrence:

$$F(n) = F(n-1) + F(n-2) \text{ and } F(0) = 0, F(1) = 1.$$

$$T(n) = O(n) + \sum_{i=1}^{n-1} T(i)$$

These numbers have many interesting and amazing properties.
A journal *The Fibonacci Quarterly*!

1. $F(n) = (\phi^n - (1-\phi)^n)/\sqrt{5}$ where ϕ is the golden ratio $(1+\sqrt{5})/2 \simeq 1.618$.
2. $\lim_{n\to\infty} F(n+1)/F(n) = \phi$

Recursive Algorithm for Fibonacci Numbers

Question: Given n, compute $F(n)$.

```
Fib(n):
    if (n = 0)
        return 0
    else if (n = 1)
        return 1
    else
        return Fib(n − 1) + Fib(n − 2)
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Running time? Let $T(n)$ be the number of additions in Fib(n).

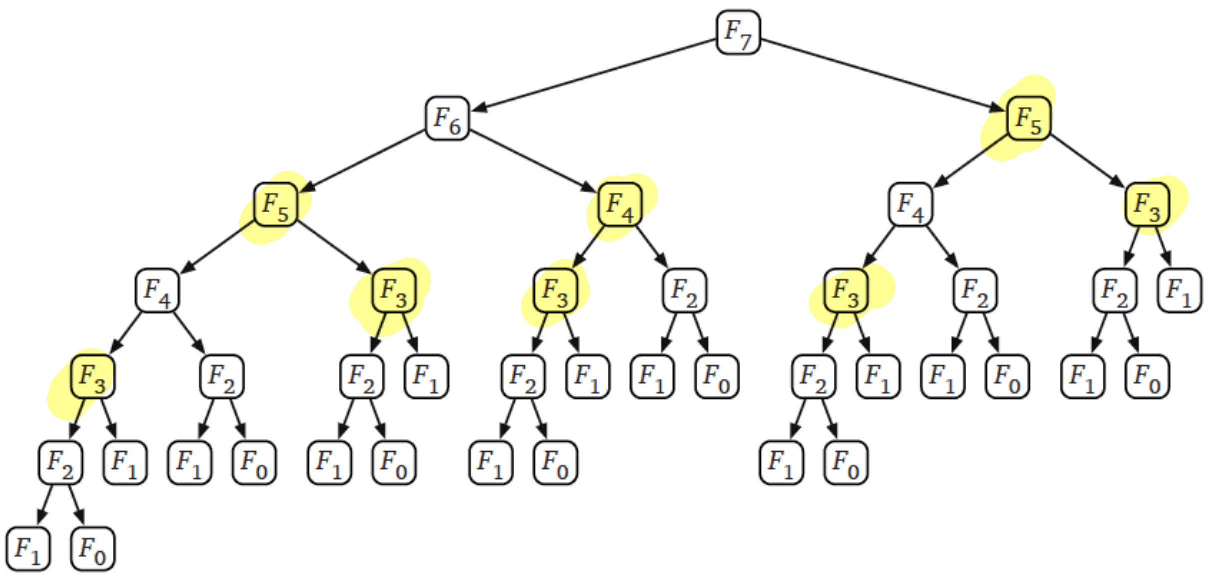
$$T(n) = T(n-1) + T(n-2) + 1 \text{ and } T(0) = T(1) = 0$$

Roughly same as $F(n)$

$$T(n) = \Theta(\phi^n)$$

The number of additions is exponential in n. Can we do better?

Recursion Tree



Memoization

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Memoization

1. Write down the results of recursive calls and look them up later
2. An array $F(n)$, where $F(i)$ stores the result of **Fib**(i)
3. Evaluation order: From bottom up, $i = 2$ then $i = 3$ and so on

An iterative algorithm for Fibonacci numbers

FibIter(n):
    if ($n = 0$) **then**
        return 0
    if ($n = 1$) **then**
        return 1
    $F[0] = 0$
    $F[1] = 1$
    for i = 2 to n **do**
        $F[i] = F[i-1] + F[i-2]$
    return $F[n]$

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    F[0] = 0
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    return F[n]
```

What is the running time of the algorithm?

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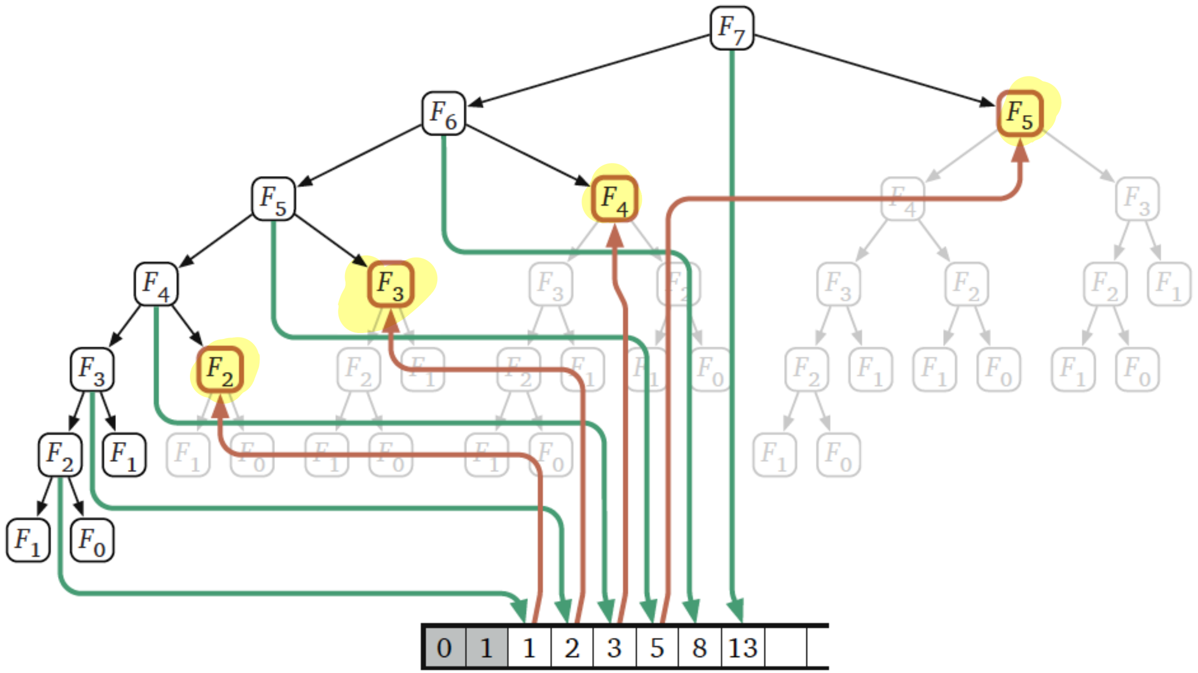
    for i = 2 to n do
        F[i] = F[i − 1] + F[i − 2]
    return F[n]
```

What is the running time of the algorithm? $O(n)$ additions.

DP prunes recursion tree



What is the difference?

Dynamic Programming:

Finding a recursion that can be *effectively/efficiently* memoized.

Leads to polynomial time algorithm if number of distinct sub-problems is polynomial in input size.

Saving space

Do we need an array of n numbers? Not really.

```
FibIter(n):
    if (n = 0) then
        return 0
    if (n = 1) then
        return 1
    prev2 = 0
    prev1 = 1
    for i = 2 to n do
        temp = prev1 + prev2
        prev2 = prev1
        prev1 = temp

    return prev1
```

Dynamic Programming

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Dynamic Programming

Dynamic Programming: Smart recursion with memoization

- **Dynamic programming is not about filling tables. It is about finding a smart recursion. First, find the correct recursion.**
- Use **memoization** to avoid recomputation of common solutions, hence optimizing running time and space.
    - First, allocate a data structure (usually an array or a multi-dimensional array that can hold values for each of the subproblems)
    - Figure out a way to order the computation of the sub-problems starting from the base case.
- Often an *iterative* algorithm with *bottom up* computation.

Part II

Text Segmentation

Problem

Input A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function **IsStrInL**$(string \quad x)$ that decides whether x is in L

Goal Decide if $w \in L^*$ using **IsStrInL**$(string \quad x)$ as a black box sub-routine

Example

Suppose L is *English* and we have a procedure to check whether a string/word is in the *English* dictionary.

- Is the string "isthisanenglishsentence" in *English**?
- Is "stampstamp" in *English**?
- Is "zibzzzad" in *English**?

Text Segmentation

Backtracking

- Changes the problem into a sequence of decision problems
- Each tries all possibilities for the current decision
- Let the recursion fairy make all remaining decisions

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Only the suffix matters.

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Base case

- zero-length string

Recursive Solution

Assume w is stored in array $A[1..n]$

```
IsStringinLstar(A[1..n]):
    If (n = 0) Output YES
    If (IsStrInL(A[1..n]))
        Output YES
    Else
        For (i = 1 to n − 1) do
            If (IsStrInL(A[1..i]) and IsStrInLstar(A[i + 1..n]))
                Output YES

    Output NO
```

Recursive Solution

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Question: How many distinct sub-problems does $\mathsf{IsStrInLstar}(A[1..n])$ generate?

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```

Question: How many distinct sub-problems does IsStrInLstar($A[1..n]$) generate? $O(n)$

Naming subproblems

After seeing that number of subproblems is $O(n)$ we name them to help us understand the structure better.

$\mathsf{ISL}(i)$: a boolean which is 1 if $A[i..n]$ is in L^*, 0 otherwise

Base case: $\mathsf{ISL}(n+1) = 1$ interpreting $A[n+1..n]$ as ϵ

Evaluate subproblems

Recursive relation:

- $\mathbf{ISL}(i) = 1$ if $\exists i < j \leq n+1$ such that ($\mathbf{ISL}(j) = 1$ and $\mathbf{IsStrInL}(A[i..(j-1)]) = 1$)
- $\mathbf{ISL}(i) = 0$ otherwise

Alternatively: $\mathbf{ISL}(i) = \max_{i<j\leq n+1} \mathbf{ISL}(j)\mathbf{IsStrInL}(A[i..(j-1]))$

$1 \cdot 1 = 1$

Evaluate subproblems

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Alternatively: $\mathbf{ISL}(i) = \max_{i<j\leq n+1} \mathbf{ISL}(j)\mathbf{IsStrInL}(A[i..(j-1]))$
Output: $\mathbf{ISL}(1)$

Iterative Algorithm

```
IsStringinLstar-Iterative(A[1..n]):
    boolean ISL[1..(n+1)]
    ISL[n+1] = TRUE
    for (i = n down to 1)
        ISL[i] = FALSE
        for (j = i+1 to n+1)
                If (ISL[j] and IsStrInL(A[i..j-1]))
                    ISL[i] = TRUE
                    Break

    If (ISL[1] = 1) Output YES
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- **Running time:**

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- **Running time:** $O(n^2)$ (assuming call to **IsStrInL** is $O(1)$ time) $1+2+\cdots+(n-1) = O(n^2)$

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- **Space:**

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- **Space:** $O(n)$

How to design DP algorithms

1. Find a "smart" recursion (The hard part)
    1. Formulate the sub-problem
    2. so that the number of distinct subproblems is small; polynomial in the original problem size.

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2. Memoization
    1. Identify distinct subproblems
    2. Choose a memoization data structure
    3. Identify dependencies and find a good evaluation order
    4. An iterative algorithm replacing recursive calls with array lookups

Part III

Longest Increasing Subsequence

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an **increasing subsequence** $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

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Example

1. Sequence: 6, 3, 5, 2, 7, 8, 1
2. Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
3. Longest increasing subsequence: 3, 5, 7, 8

Recursive Approach: Take 1

LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

$\mathrm{LIS}(A[1..n])$:

1. Case 1: Does not contain $A[n]$ in which case $\mathrm{LIS}(A[1..n]) = \mathrm{LIS}(A[1..(n-1)])$
2. Case 2: contains $A[n]$ in which case $\mathrm{LIS}(A[1..n])$ is not so clear.

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Observation

*For second case we want to find a subsequence in $A[1..(n-1)]$ that is restricted to numbers less than $A[n]$. This suggests that a more general problem is **LIS_smaller**$(A[1..n], x)$ which gives the longest increasing subsequence in A where each number in the sequence is less than x.*

Recursive Approach

$LIS(A[1..n])$: the length of longest increasing subsequence in A

LIS_smaller$(A[1..n], x)$: length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than x

```
LIS_smaller(A[1..n], x):
    if (n = 0) then return 0
    m = LIS_smaller(A[1..(n − 1)], x)
    if (A[n] < x) then
        m = max(m, 1 + LIS_smaller(A[1..(n − 1)], A[n]))
    Output m
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LIS(A[1..n]):
    return LIS_smaller(A[1..n], ∞)
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- How much space for memoization?

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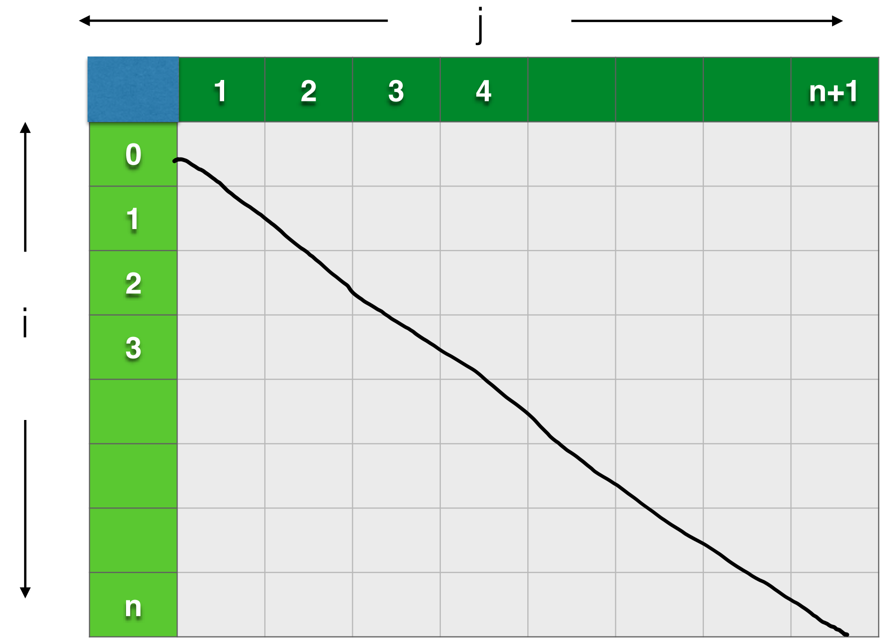
- How much space for memoization? $O(n^2)$

Naming subproblems and recursive equation

After seeing that number of subproblems is $O(n^2)$ we name them to help us understand the structure better. For notational ease we add ∞ at end of array (in position $n+1$)

$\mathrm{LIS}(i, j)$: length of longest increasing sequence in $A[1..i]$ among numbers less than $A[j]$ (defined only for $i < j$)

How to order bottom up computation?



How to order bottom up computation?

j

i

	1	2	3	4				n+1
0	0	0	0	0	0	0	0	0
1								
2								
3								
n								

Base case: $\mathrm{LIS}(0, j) = 0$ for $1 \leq j \leq n + 1$

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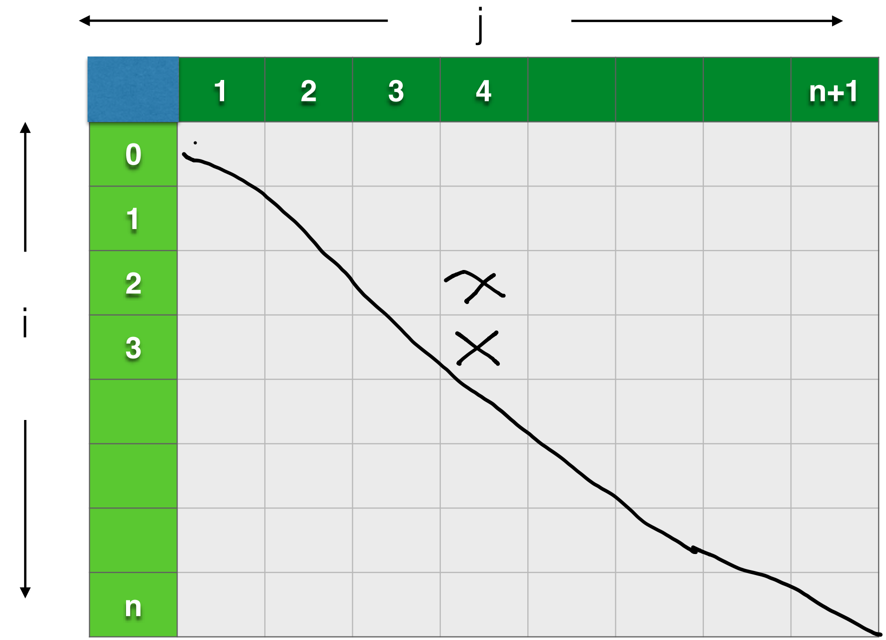

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Recursive relation:

- $\mathrm{LIS}(i, j) = \mathrm{LIS}(i-1, j)$ if $A[i] > A[j]$
- $\mathrm{LIS}(i, j) = \max\{LIS(i-1, j), 1 + LIS(i-1, i)\}$ if $A[i] \leq A[j]$

How to order bottom up computation?

Sequence: $A[1..7] = 6, 3, 5, 2, 7, 8, 1$

6 3 5 2 7 8 1 ∞

j

6 3 5 2 7 8 1

i

	1	2	3	4				n+1
0	0	0	0	0	0	0	0	0
1		0	0	0	1	1	0	1
2			1	0	1	1	0	1
3				0	2	2	0	2
					2	2	0	2
						3	0	3
							0	4
n								4

Iterative algorithm

```
LIS-Iterative(A[1..n]):
    A[n + 1] = ∞
    int LIS[0..n, 1..n + 1]
    for (j = 1 to n + 1) do
        LIS[0, j] = 0

    for (i = 1 to n) do
        for (j = i + 1 to n)
            If (A[i] > A[j]) LIS[i, j] = LIS[i − 1, j]
            Else LIS[i, j] = max{LIS[i − 1, j], 1 + LIS[i − 1, i]}

    Return LIS[n, n + 1]
```

Running time: $O(n^2)$

Space: $O(n^2)$