Backtracking

Lecture 12

Recursion types

1. **Divide and Conquer**: Problem reduced to multiple **independent** sub-problems.

   Examples: Merge sort, quick sort, multiplication, median selection.

2. **Backtracking**

Part I

N Queens Problem

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Definition

Place n queens on an $n \times n$ board so that no two queens are attacking each other.

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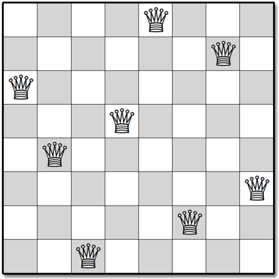
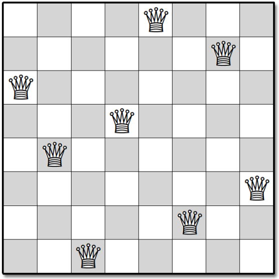


N Queens Problem



Brute-force algorithm:

Try all combinations of n positions.

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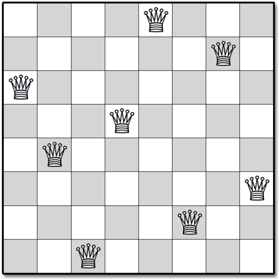

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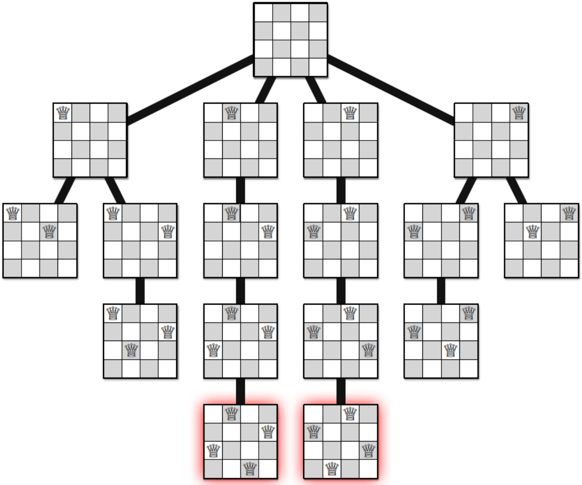
Try all combinations of n positions.

Methodical brute-force:

No two queens on the same row, so place a queen in one row at a time.

N Queens Problem



N Queens Problem

Base case

- when any position in the row is attacked by a queen on an earlier row, recursion terminates.
- Or when all n queens are placed.

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- Changes the problem into a sequence of decision problems
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- Let the recursion fairy make all remaining decisions

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How do we redefine the problem to make recursion work?

- The recursion does not solve the $n - 1$ queens problem
- We need to place the r-th queen so that it is not attacked by a queen on an earlier row
- The recursive subproblem:
  - Input $= r - 1$ queens placed in earlier rows
  - Place the remaining $n - r + 1$ queens, one on each row
  - Recurse by increasing r

N Queens Problem

```
PLACEQUEENS(Q[1..n], r):
  if r = n + 1
      print Q[1..n]
  else
      for j ← 1 to n
          legal ← TRUE
          for i ← 1 to r − 1
              if (Q[i] = j) or (Q[i] = j + r − i) or (Q[i] = j − r + i)
                  legal ← FALSE
          if legal
              Q[r] ← j
              PLACEQUEENS(Q[1..n], r + 1)      ⟨⟨Recursion!⟩⟩
```

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   Each subproblem is only a constant smaller, e.g. from n to $n-1$.
   e.g. $T(n) = n \cdot T(n-1)$, $T(1) = n$, hence $T(n) = O(n^n)$.
   e.g. $T(n) = 2 \cdot T(n-1) + O(1)$, hence $T(n) = O(2^n)$.

Part II

Text Segmentation

Problem

Input: A string $w \in \Sigma^*$ and access to a language $L \subseteq \Sigma^*$ via function **IsStrInL**$(string\ \ x)$ that decides whether x is in L

Goal: Decide if $w \in L^*$ using **IsStrInL**$(string\ \ x)$ as a black box sub-routine

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Example

Suppose L is *English* and we have a procedure to check whether a string/word is in the *English* dictionary.

- Is the string "isthisanenglishsentence" in *English**?
- Is "stampstamp" in *English**?
- Is "zibzzzad" in *English**?

Text Segmentation

Backtracking

- Changes the problem into a sequence of decision problems

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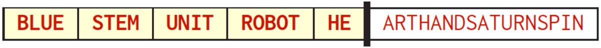

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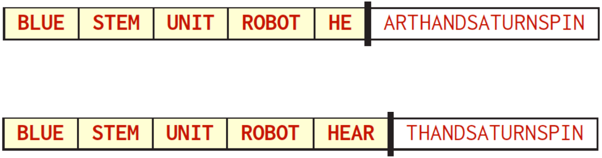

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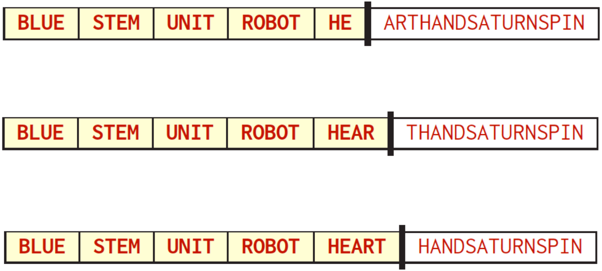

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Text segmentation

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Base case

- zero-length string

Recursive Solution

Assume w is stored in array $A[1..n]$

```
IsStringinLstar(A[1..n]):
    If (n = 0) Output YES
    If (IsStrInL(A[1..n]))
        Output YES
    Else
        For (i = 1 to n − 1) do
            If (IsStrInL(A[1..i]) and IsStrInLstar(A[i + 1..n]))
                Output YES

    Output NO
```

Part III

Longest Increasing Subsequence

Sequences

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Sequence: an ordered list $a_1, a_2, \ldots, a_n$. **Length** of a sequence is number of elements in the list.

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$1 \leq i_1 < i_2 < \ldots < i_k \leq n$.

Definition

A sequence is **increasing** if $a_1 < a_2 < \ldots < a_n$. It is **non-decreasing** if $a_1 \leq a_2 \leq \ldots \leq a_n$. Similarly **decreasing** and **non-increasing**.

Sequences

Example...

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1. Sequence: $6, 3, 5, 2, 7, 8, 1, 9$
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1. Sequence: $6, 3, 5, 2, 7, 8, 1, 9$
2. Subsequence of above sequence: $5, 2, 1$
3. Increasing sequence: $3, 5, 9, 17, 54$
4. Decreasing sequence: $34, 21, 7, 5, 1$
5. Increasing subsequence of the first sequence: $2, 7, 9$.

Longest Increasing Subsequence Problem

Input A sequence of numbers $a_1, a_2, \ldots, a_n$

Goal Find an **increasing subsequence** $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ of maximum length

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Example

1. Sequence: 6, 3, 5, 2, 7, 8, 1
2. Increasing subsequences: 6, 7, 8 and 3, 5, 7, 8 and 2, 7 etc
3. Longest increasing subsequence: 3, 5, 7, 8

Recursive Approach: Take 1

LIS: Longest increasing subsequence

Can we find a recursive algorithm for LIS?

LIS($A[1..n]$):

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Observation

For second case we want to find a subsequence in $A[1..(n-1)]$ *that is restricted to numbers less than* $A[n]$.

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Observation

For second case we want to find a subsequence in $A[1..(n-1)]$ that is restricted to numbers less than $A[n]$. This suggests that a more general problem is **LIS_smaller**($A[1..n], x$) *which gives the longest increasing subsequence in A where each number in the sequence is less than x.*

Recursive Approach

LIS_smaller$(A[1..n], x)$ **:** length of longest increasing subsequence in $A[1..n]$ with all numbers in subsequence less than x

```
LIS_smaller(A[1..n], x):
    if (n = 0) then return 0
    m = LIS_smaller(A[1..(n − 1)], x)
    if (A[n] < x) then
        m = max(m, 1 + LIS_smaller(A[1..(n − 1)], A[n]))
    Output m
```

```
LIS(A[1..n]):
        return LIS_smaller(A[1..n], ∞)
```

Part IV

From Backtracking to DP

Running time analysis of Text Segmentation

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IsStringinLstar(A[1..n]):
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Running time: $O(2^n)$

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However, how many suffixes are there? $O(n)$

Different past decision can lead to the same suffix.

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