CS/ECE 374: Algorithms & Models of Computation

Recursion

Lecture 10

We will learn

1. How to ask the recursion fairy to solve the problem for us.

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2. How to analyze the running time of a recursive algorithm.

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2. How to analyze the running time of a recursive algorithm.
3. Recursion in action
    1. Tower of Hanoi puzzle
    2. Merge sort
    3. Quick sort

Recursion

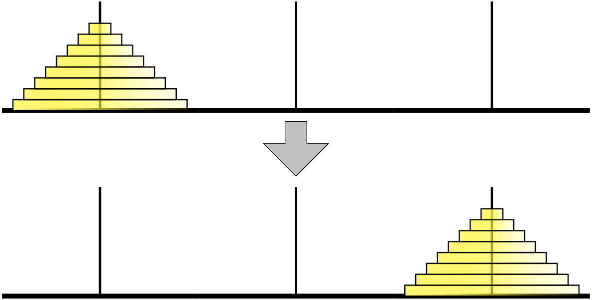
How to think about it

Recursion = Induction

Part I

Tower of Hanoi

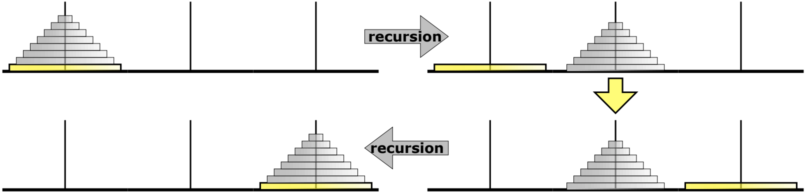
Tower of Hanoi



The Tower of Hanoi puzzle

Move stack of ***n*** disks from peg **0** to peg **2**, one disk at a time.
Rule: cannot put a larger disk on a smaller disk.
Question: what is a strategy and how many moves does it take?

Tower of Hanoi via Recursion



The Tower of Hanoi algorithm; ignore everything but the bottom disk

Recursive Algorithm

```
Hanoi($n$, src, dest, tmp):
    Hanoi($n-1$, src, tmp, dest)
    Move disk $n$ from src to dest
    Hanoi($n-1$, tmp, dest, src)
```

Recursive Algorithm

```
Hanoi($n$, src, dest, tmp):
    if ($n > 0$) then
        Hanoi($n - 1$, src, tmp, dest)
        Move disk $n$ from src to dest
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Proof of correctness.

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Running time analysis.
$T(n)$: time to move n disks via recursive strategy

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Running time analysis.
$T(n)$: time to move n disks via recursive strategy

$$T(n) = 2T(n-1) + 1 \qquad n > 1 \qquad \text{and } T(1) = 1$$

Analysis

$$\begin{aligned}
T(n) &= 2T(n-1)+1 \\
&= 2^2T(n-2)+2+1 \\
&= \ldots \\
&= 2^iT(n-i)+2^{i-1}+2^{i-2}+\ldots+1 \\
&= \ldots \\
&= 2^{n-1}T(1)+2^{n-2}+\ldots+1 \\
&= 2^{n-1}+2^{n-2}+\ldots+1 \\
&= (2^n-1)/(2-1)=2^n-1
\end{aligned}$$

Part II

Merge Sort

Sorting

Input	Given an array of n elements
Goal	Rearrange them in ascending order

Merge Sort [von Neumann]

MergeSort

1. Input: Array $A[1 \ldots n]$

$$A \; L \; G \; O \; R \; I \; T \; H \; M \; S$$

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1. Input: Array $A[1 \dots n]$

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2. Divide into subarrays $A[1 \dots m]$ and $A[m+1 \dots n]$, where $m = \lfloor n/2 \rfloor$

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3. Recursively **MergeSort** $A[1 \dots m]$ and $A[m+1 \dots n]$

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4. Merge the sorted arrays

   $$A\ G\ H\ I\ L\ M\ O\ R\ S\ T$$

Merging Sorted Arrays

1. Use a new array B to store the merged array
2. Scan $A[1 \dots m]$ and $A[m+1 \dots n]$ from left-to-right, storing elements in B in order

$$\begin{array}{ccccccccccc} A & G & L & O & R & \quad & H & I & M & S & T \\ A & & & & & & & & & & \end{array}$$

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$$A\ G$$

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Formal Code

```
MERGESORT($A[1..n]$):
  if $n > 1$
      $m \leftarrow \lfloor n/2 \rfloor$
      MERGESORT($A[1..m]$)
      MERGESORT($A[m+1..n]$)
      MERGE($A[1..n], m$)
```

```
MERGE($A[1..n], m$):
  $i \leftarrow 1$;  $j \leftarrow m+1$
  for $k \leftarrow 1$ to $n$
      if $j > n$
          $B[k] \leftarrow A[i]$;  $i \leftarrow i+1$
      else if $i > m$
          $B[k] \leftarrow A[j]$;  $j \leftarrow j+1$
      else if $A[i] < A[j]$
          $B[k] \leftarrow A[i]$;  $i \leftarrow i+1$
      else
          $B[k] \leftarrow A[j]$;  $j \leftarrow j+1$

  for $k \leftarrow 1$ to $n$
      $A[k] \leftarrow B[k]$
```

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- How do we prove that Merge is correct? Also by induction!
- One way is to rewrite Merge into a recursive version.
- For algorithms with loops one comes up with a natural *loop invariant* that captures all the essential properties and then we prove the loop invariant by induction on the index of the loop.

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- For algorithms with loops one comes up with a natural *loop invariant* that captures all the essential properties and then we prove the loop invariant by induction on the index of the loop.

At the start of iteration k the following hold:

- $B[1..k]$ contains the smallest k elements of A correctly sorted.
- $B[1..k]$ contains the elements of $A[1..(i-1)]$ and $A[(m+1)..(j-1)]$.
- No element of A is modified.

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What do we want as a solution to the recurrence?

Almost always only an *asymptotically* tight bound. That is we want to know $f(n)$ such that $T(n) = \Theta(f(n))$.

1. $T(n) = O(f(n))$ - upper bound
2. $T(n) = \Omega(f(n))$ - lower bound

Solving Recurrences: Some Techniques

1. Know some basic math: geometric series, logarithms, exponentials, elementary calculus
2. Expand the recurrence and spot a pattern and use simple math
3. Recursion tree method — imagine the computation as a tree
4. Guess and verify — useful for proving upper and lower bounds even if not tight bounds

Recursion Trees

Part III

Quick Sort

Quick Sort

Quick Sort [Hoare]

1. Pick a pivot element from array
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.

3. Recursively sort the subarrays, and concatenate them.

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Quick Sort: Example

1. array: 16, 12, 14, 20, 5, 3, 18, 19, 1
2. pivot: 16

Time Analysis

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    1. Theoretically, median can be found in linear time.
3. Typically, pivot is the first or last element of array. Then,

$$T(n) = \max_{1 \leq k \leq n} (T(k-1) + T(n-k) + O(n))$$

In the worst case $T(n) = T(n-1) + O(n)$, which means $T(n) = O(n^2)$. Happens if array is already sorted and pivot is always first element.

Recursion Trees