Assume $L$ is any regular language. Let’s define a new language:

**Definition**

$$\text{Flip}(L) = \{ \bar{w} \mid w \in L, x \in \Sigma^* \}$$
CS/ECE-374: Lecture 6 - Regular Languages - Closure Properties

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Pre-lecture brain teaser

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Yes
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**Definition**

$\text{Flip}(L) = \{ \bar{w} \mid w \in L, x \in \Sigma^* \}$

Yes Next problem.
Assume $L$ is any regular language. Let’s define a new language:

**Definition**

$L^R = \{w^R \mid w \in L\}$
Assume $L$ is any regular language. Let's define a new language:

**Definition**

$L^R = \{w^R | w \in L\}$

Also yes.
Definition
(Informal) A set $A$ is **closed** under an operation $op$ if applying $op$ to any elements of $A$ results in an element that also belongs to $A$. 
Closure properties

Definition
(Informal) A set $A$ is closed under an operation $\text{op}$ if applying $\text{op}$ to any elements of $A$ results in an element that also belongs to $A$.

Examples:

- **Integers**: closed under $+$, $-$, $\ast$, but not division.
- **Positive integers**: closed under $+$ but not under $-$
- **Regular languages**: closed under union, intersection, Kleene star, complement, difference, homomorphism, inverse homomorphism, reverse, ...
How do we prove that regular languages are closed under some new operation?
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Three broad approaches

- Use existing closure properties
Closure properties of Regular Languages

How do we prove that regular languages are closed under some new operation?

Three broad approaches

• Use existing closure properties
  • $L_1, L_2, L_3, L_4$ regular implies $(L_1 - L_2) \cap (\overline{L_3 \cup L_4})^*$ is regular
How do we prove that regular languages are closed under some new operation?

Three broad approaches

- Use existing closure properties
  - $L_1, L_2, L_3, L_4$ regular implies $(L_1 - L_2) \cap (L_3 \cup L_4)^*$ is regular
- Transform regular expressions
Closure properties of Regular Languages

How do we prove that regular languages are closed under some new operation?

Three broad approaches

- Use existing closure properties
  - $L_1, L_2, L_3, L_4$ regular implies $(L_1 - L_2) \cap (\overline{L_3} \cup L_4)^*$ is regular
- Transform regular expressions
- Transform DFAs to NFAs — versatile technique and shows the power of nondeterminism
Let’s look back at the pre-lecture teaser. Define a function

\[ h(x) = \begin{cases} 
1 & x = 0 \\
0 & x = 1 
\end{cases} \]

This is known as a homomorphism - A cipher that is a one-to-one mapping to one character set to another.

How do we prove \( h(L) \) is regular if \( L \) is regular?
Homomorphism closure

Proof Idea:

1. Suppose $R$ is a regular expression for $L$.
2. We define $Flip(L) = L^F$ as a regular expression based off the regular expression for $L$ (using a finite number of concatenations, unions and Kleene Star).
3. Thus $L^F$ is regular because it has a regular expression.

Thus we reduce the argument to $L(h(R)) = h(L(R))$
Homomorphism closure

Let’s define the regular expression inductively by transforming the operations in $R$. We see that:

- **Base Case:** Zero operators in $R$ means that $R =: a \in \Sigma, \varepsilon, \emptyset$. In any case we define $R^F = h(R)$
- Otherwise $R$ has three potential types of operators to transform. Splitting $R$ at an operator we see:
  - $h(R_1R_2) = h(R_1) \cdot h(R_2)$
  - $h(R_1 \cup R_2) = h(R_1) \cup h(R_2)$
  - $h(R^*) = (h(R))^*$

Hence, since we can define $L^F$ via a regular language, $L^F$ is regular.
Regular Languages

Regular languages have three different characterizations

- Inductive definition via base cases and closure under union, concatenation and Kleene star
- Languages accepted by DFAs
- Languages accepted by NFAs
Regular Languages

Regular languages have three different characterizations

- Inductive definition via base cases and closure under union, concatenation and Kleene star
- Languages accepted by DFAs
- Languages accepted by NFAs

Regular language closed under many operations:

- union, concatenation, Kleene star via inductive definition or NFAs
- complement, union, intersection via DFAs
- homomorphism, inverse homomorphism, reverse, …

Different representations allow for flexibility in proofs.
Closure problem - Reverse
Example: REVERSE

Given string \( w \), \( w^R \) is reverse of \( w \).

For a language \( L \) define \( L^R = \{ w^R \mid w \in L \} \) as reverse of \( L \).

**Theorem**

\( L^R \) is regular if \( L \) is regular.
Example: REVERSE

Given string $w$, $w^R$ is reverse of $w$.

For a language $L$ define $L^R = \{w^R \mid w \in L\}$ as reverse of $L$.

**Theorem**

$L^R$ is regular if $L$ is regular.

Infinitely many regular languages!

Proof technique:

- take some finite representation of $L$ such as regular expression $r$
- Describe an algorithm $A$ that takes $r$ as input and outputs a regular expression $r'$ such that $L(r') = (L(r))^R$.
- Come up with $A$ and prove its correctness.
Suppose $r$ is a regular expression for $L$. How do we create a regular expression $r'$ for $L^R$?
Suppose $r$ is a regular expression for $L$. How do we create a regular expression $r’$ for $L^R$? Inductively based on recursive definition of $r$.

- $r = \emptyset$ or $r = a$ for some $a \in \Sigma$
- $r = r_1 + r_2$
- $r = r_1 \cdot r_2$
- $r = (r_1)^*$
REVERSE via regular expressions

- $r = \emptyset$ or $r = a$ for some $a \in \Sigma$
  
  $r' = \emptyset$

- $r = r_1 + r_2$.
  
  If $r'_1, r'_2$ are reg expressions for $(L(r_1))^R, (L(r_2))^R$ then
  
  $r' = r'_1 + r'_2$

- $r = r_1 \cdot r_2$.
  
  If $r'_1, r'_2$ are reg expressions for $(L(r_1))^R, (L(r_2))^R$ then
  
  $r' = r'_1 \cdot r'_2$

- $r = (r_1)^*$.
  
  If $r'_1$ is reg expressions for $(L(r_1))^R$ then
  
  $r' = (r'_1)^*$

$r = (0 + 10)^*(001 + 01)1$ then $r' =$
REVERSE via machine transformation

Given DFA $M = (Q, \Sigma, \delta, s, A)$ want NFA $N$ such that $L(N) = (L(M))^R$.

$N$ should accept $w^R$ iff $M$ accepts $w$

$M$ accepts $w$ iff $\delta_M^*(s, w) \in A$

Idea:
Caveat: Reversing transitions may create an NFA.
REVERSE via machine transformation

Proof (DFA to NFA): Let $M = (\Sigma, Q, s, A, \delta)$ be an arbitrary DFA that accepts $L$. We construct an NFA $M^R = (\Sigma, Q^R, s^R, A^R, \delta^R)$ with \( \epsilon \)-transitions that accepts $L^R$, intuitively by reversing every transition in $M$, and swapping the roles of the start state and the accepting states. Because $M$ does not have a unique accepting state, we need to introduce a special start state $s^R$, with \( \epsilon \)-transitions to each accepting state in $M$. These are the only \( \epsilon \)-transitions in $M^R$.

$$Q^R = Q \cup \{s^R\}$$
$$A^R = \{s\}$$
$$\delta^R(s^R, \epsilon) = A$$
$$\delta^R(s^R, a) = \emptyset$$
$$\delta^R(q, \epsilon) = \emptyset$$
$$\delta^R(q, a) = \{p \mid q \in \delta(p, a)\}$$

for all $a \in \Sigma$

for all $q \in Q$

for all $q \in Q$ and $a \in \Sigma$

Routine inductive definition-chasing now implies that the reversal of any sequence $q_0 \rightarrow q_1 \rightarrow \cdots \rightarrow q_\ell$ of transitions in $M$ is a valid sequence $q_\ell \rightarrow q_{\ell-1} \rightarrow \cdots \rightarrow q_0$ of transitions in $M^R$. Because the transitions retain their labels (but reverse directions), it follows that $M$ accepts any string $w$ if and only if $M^R$ accepts $w^R$.

We conclude that the NFA $M^R$ accepts $L^R$, so $L^R$ must be regular. \(\square\)
Formal proof: two directions

- \( w \in L(M) \) implies \( w^R \in L(N) \). Sketch. Let \( \delta^*_M(s, w) = q \) where \( q \in A \). On input \( w^R \) \( N \) non-deterministically transitions from its start state \( s' \) to \( q \) on an \( \epsilon \) transition, and traces the reverse of the walk of \( M \) on \( w^R \) and hence reaches \( s \) which is an accepting state of \( N \). Thus \( N \) accepts \( w^R \)

- \( u \in L(N) \) implies \( u^R \in L(M) \). Sketch. If \( u \in N \) it implies that \( s' \) transitioned to some \( q \in A \) on \( \epsilon \) transition and
Closure Problem - Cycle
A more complicated example: CYCLE

\[ CYCLE(L) = \{yx \mid x, y \in \Sigma^*, xy \in L \} \]

**Theorem**

*CYCLE(L)* is regular if *L* is regular.

**Example:** *L* = \{*abc*, *374a*\}

\[ CYCLE(L) = \]
A more complicated example: CYCLE

\[
CYCLE(L) = \{yx \mid x, y \in \Sigma^*, xy \in L\}
\]

**Theorem**

\(CYCLE(L)\) is regular if \(L\) is regular.
A more complicated example: CYCLE

\[ CYCLE(L) = \{yx \mid x, y \in \Sigma^* \text{, } xy \in L \} \]

**Theorem**

*CYCLE(L) is regular if L is regular.*

Given DFA \( M \) for \( L \) create NFA \( N \) that accepts \( CYCLE(L) \).

- \( N \) is a finite state machine, cannot know split of \( w \) into \( xy \) and yet has to simulate \( M \) on \( x \) and \( y \).
- Exploit fact that \( M \) is itself a finite state machine. \( N \) only needs to “know” the state \( \delta^*_M(s, x) \) and there are only finite number of states in \( M \).
Construction for CYCLE

Let \( w = xy \) and \( w' = yx \).

- \( N \) guesses state \( q = \delta^*_M(s, x) \) and simulates \( M \) on \( w' \) with start state \( q \).
- \( N \) guesses when \( y \) ends (at that point \( M \) must be in an accept state) and transitions to a copy of \( M \) to simulate \( M \) on remaining part of \( w' \) (which is \( x \))
- \( N \) accepts \( w' \) if after second copy of \( M \) on \( x \) it ends up in the guessed state \( q \)
Construction for CYCLE
Exercise: Write down formal description of $N$ in tuple notation starting with $M = (Q, \Sigma, \delta, s, A)$.

Need to argue that $L(N) = CYCLE(L(M))$

- If $w = xy$ accepted by $M$ then argue that $yx$ is accepted by $N$
- If $N$ accepts $w'$ then argue that $w' = yx$ such that $xy$ accepted by $M$. 
Closure Problem - Prefix
Let \( L \) be a language over \( \Sigma \).

**Definition**

\[
\text{PREFIX}(L) = \{ w \mid wx \in L, x \in \Sigma^* \}
\]
Example: PREFIX

Let $L$ be a language over $\Sigma$.

**Definition**
$\text{PREFIX}(L) = \{ w \mid wx \in L, x \in \Sigma^* \}$

**Theorem**
*If $L$ is regular then $\text{PREFIX}(L)$ is regular.*
Example: PREFIX

Let $L$ be a language over $\Sigma$.

**Definition**

$\text{PREFIX}(L) = \{ w \mid wx \in L, x \in \Sigma^* \}$

**Theorem**

*If $L$ is regular then $\text{PREFIX}(L)$ is regular.*

Let $M = (Q, \Sigma, \delta, s, A)$ be a DFA that recognizes $L$.
Example: PREFIX

Let $L$ be a language over $\Sigma$.

**Definition**

$\text{PREFIX}(L) = \{ w \mid wx \in L, x \in \Sigma^* \}$

**Theorem**

*If $L$ is regular then $\text{PREFIX}(L)$ is regular.*

Let $M = (Q, \Sigma, \delta, s, A)$ be a DFA that recognizes $L$

$X = \{ q \in Q \mid s \text{ can reach } q \text{ in } M \}$
Let $L$ be a language over $\Sigma$.

**Definition**

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**Theorem**

*If $L$ is regular then $\text{PREFIX}(L)$ is regular.*

Let $M = (Q, \Sigma, \delta, s, A)$ be a DFA that recognizes $L$

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$Y = \{q \in Q \mid q \text{ can reach some state in } A\}$
Example: PREFIX

Let \( L \) be a language over \( \Sigma \).

**Definition**

\[ \text{PREFIX}(L) = \{ w \mid wx \in L, x \in \Sigma^* \} \]

**Theorem**

*If \( L \) is regular then \( \text{PREFIX}(L) \) is regular.*

Let \( M = (Q, \Sigma, \delta, s, A) \) be a DFA that recognizes \( L \)

\[ X = \{ q \in Q \mid s \text{ can reach } q \text{ in } M \} \]

\[ Y = \{ q \in Q \mid q \text{ can reach some state in } A \} \]

\[ Z = X \cap Y \]

Create new DFA \( M' = (Q, \Sigma, \delta, s, Z) \)
Let $L$ be a language over $\Sigma$.

**Definition**

$\text{PREFIX}(L) = \{ w | wx \in L, x \in \Sigma^* \}$

**Theorem**

*If $L$ is regular then $\text{PREFIX}(L)$ is regular.*

Let $M = (Q, \Sigma, \delta, s, A)$ be a DFA that recognizes $L$.

$X = \{ q \in Q | s \text{ can reach } q \text{ in } M \}$

$Y = \{ q \in Q | q \text{ can reach some state in } A \}$

$Z = X \cap Y$

Create new DFA $M' = (Q, \Sigma, \delta, s, Z)$

**Claim:** $L(M') = \text{PREFIX}(L)$. 

Exercise: SUFFIX

Let $L$ be a language over $\Sigma$.

**Definition**

$\text{SUFFIX}(L) = \{w \mid xw \in L, x \in \Sigma^*\}$

Prove the following:

**Theorem**

*If $L$ is regular then $\text{SUFFIX}(L)$ is regular.*
Let $L$ be a language over $\Sigma$.

**Definition**
$\text{SUFFIX}(L) = \{ w \mid xw \in L, x \in \Sigma^* \}$

Prove the following:

**Theorem**
*If $L$ is regular then $\text{SUFFIX}(L)$ is regular.*

Same idea as $\text{PREFIX}(L)$

$X = \{ q \in Q \mid s \text{ can reach } q \text{ in } M \}$

$Y = \{ q \in Q \mid q \text{ can reach some state in } A \}$

$Z = X \cap Y$

With one major **difference**:
We can also prove non-regularity using the techniques above. For instance:

\[ L_1 = \{0^n 1^n | n \geq 0\} \]

\[ L_2 = \{w \in \{0, 1\}^* | \#0(w) = \#1(w)\} \]

\[ L_3 = \{0^i 1^j | i \neq j\} \]

\[ L_1 \] is not regular. Can we use that fact to prove \[ L_2 \] and \[ L_3 \] are not regular without going through the fooling set argument?

\[ L_1 = \overline{L_3} \cap 0^* 1^* \]

Hence if \[ L_3 \] is regular then \[ L_1 \] is regular, a contradiction.
Application of closure properties to non-regularity

We can also prove non-regularity using the techniques above. For instance:

\[ L_1 = \{0^n1^n \mid n \geq 0\} \]

\[ L_2 = \{w \in \{0, 1\}^* \mid \#_0(w) = \#_1(w)\} \]

\[ L_3 = \{0^i1^j \mid i \neq j\} \]

\[ L_1 \text{ is not regular. Can we use that fact to prove } L_2 \text{ and } L_3 \text{ are not regular without going through the fooling set argument?} \]

\[ L_1 = \overline{L_3} \cap 0^*1^* \text{ hence if } L_3 \text{ is regular then } L_1 \text{ is regular, a contradiction.} \]
We can also prove non-regularity using the techniques above. For instance:

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\( L_1 = L_2 \cap 0^*1^* \) hence if \( L_2 \) is regular then \( L_1 \) is regular, a contradiction.
We can also prove non-regularity using the techniques above. For instance:

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\( L_1 \) is not regular. Can we use that fact to prove \( L_2 \) and \( \bar{L}_2 \) are not regular without going through the fooling set argument?

\( L_1 = L_2 \cap 0^*1^* \) hence if \( L_2 \) is regular then \( L_1 \) is regular, a contradiction.

\( L_1 = \bar{L}_3 \cap 0^*1^* \) hence if \( L_3 \) is regular then \( L_1 \) is regular, a contradiction.