Pre-lecture brain teaser

Consider the problem of a n-input *AND* function. The input (x) is a string n-digits long with $\Sigma = \{0, 1\}$ and has an output (y) which is the logical *AND* of all the elements of x.

Formulate a **language** that describes the above problem.

CS/ECE-374: Lecture 2 - Regular Languages

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University of Illinois at Urbana Champaign

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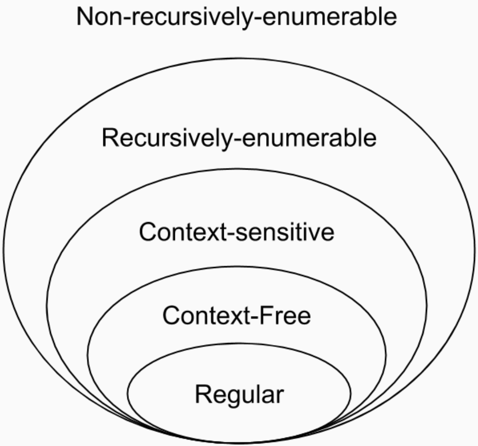
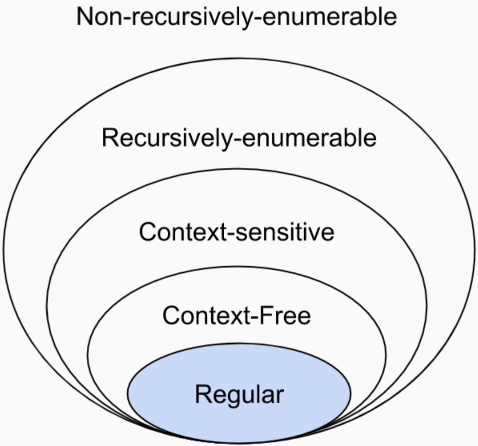
This is an example of a regular language which we'll be discussing today.

Chomsky Hierarchy



Chomsky Hierarchy



Regular Languages

Regular Languages

Theorem (Kleene's Theorem)

A language is regular if and only if it can be obtained from finite languages by applying the three operations:

- *Union*
- *Concatenation*
- *Repetition*

a finite number of times.

Regular Languages

A class of simple but useful languages.

The set of regular languages over some alphabet Σ is defined inductively.

Base Case

- $\emptyset$ is a regular language.
- $\{\epsilon\}$ is a regular language.
- $\{a\}$ is a regular language for each $a \in \Sigma$. Interpreting a as string of length 1.

Regular Languages

Inductive step:

We can build up languages using a few basic operations:

- If L_1, L_2 are regular then $L_1 \cup L_2$ is regular.
- If L_1, L_2 are regular then $L_1 L_2$ is regular.
- If L is regular, then $L^* = \cup_{n \geq 0} L^n$ is regular.
  The $\cdot^*$ operator name is *Kleene star*.
- If L is regular, then so is $\overline{L} = \Sigma^* \setminus L$.

Regular languages are closed under operations of union, concatenation and Kleene star.

Regular Languages

Have basic operations to build regular languages.

Important: Any language generated by a finite sequence of such operations is regular.

Lemma
Let $L_1, L_2, \ldots,$ be regular languages over alphabet Σ. Then the language $\cup_{i=1}^{\infty} L_i$ is not necessarily regular.

Example:

Some simple regular languages

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If w is a string then $L = \{w\}$ is regular.

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Lemma
Every finite language L is regular.

Examples: $L = \{a, abaab, aba\}$. $L = \{w \mid |w| \leq 100\}$. Why?

Rapid-fire questions - regular languages

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3. $L_3 = \left\{ 0^i \;\middle|\; i \text{ is divisible by } 2, 3, \text{or } 5 \right\}$. L_3 is regular. T/F?

Rapid-fire questions - regular languages

4. $L_4 = \{w \in \{0,1\}^* \mid w \text{ has at most 374 1s}\}$. L_4 is regular. T/F?

Regular Expressions

Regular Expressions

A way to denote regular languages

- simple patterns to describe related strings
- useful in
  - text search (editors, Unix/grep, emacs)
  - compilers: lexical analysis
  - compact way to represent interesting/useful languages
  - dates back to 50's: Stephen Kleene
    who has a star names after him [1].

[1] Kleene, Stephen C.: "Representation of Events in Nerve Nets and Finite Automata". In Shannon, Claude E.; McCarthy, John. Automata Studies, Princeton University Press. pp. 3–42., 1956.

Inductive Definition

A regular expression $\mathbf{r}$ over an alphabet Σ is one of the following:

Base cases:

- $\emptyset$ denotes the language $\emptyset$
- ϵ denotes the language $\{\epsilon\}$.
- a denote the language $\{a\}$.

Inductive cases: If $\mathbf{r_1}$ and $\mathbf{r_2}$ are regular expressions denoting languages R_1 and R_2 respectively then,

- $(\mathbf{r_1} + \mathbf{r_2})$ denotes the language $R_1 \cup R_2$
- $(\mathbf{r_1} \bullet \mathbf{r_2}) = r_1 \bullet r_2 = (\mathbf{r_1 r_2})$ denotes the language $R_1 R_2$
- $(\mathbf{r_1})^*$ denotes the language R_1^*

Regular Languages vs Regular Expressions

Regular Languages	Regular Expressions
$\emptyset$ regular	$\emptyset$ denotes $\emptyset$
$\{\epsilon\}$ regular	ϵ denotes $\{\epsilon\}$
$\{a\}$ regular for $a \in \Sigma$	**a** denote $\{a\}$
$R_1 \cup R_2$ regular if both are	$\mathbf{r_1} + \mathbf{r_2}$ denotes $R_1 \cup R_2$
$R_1 R_2$ regular if both are	$\mathbf{r_1} \bullet \mathbf{r_2}$ denotes $R_1 R_2$
R^* is regular if R is	$\mathbf{r}^*$ denote R^*

Regular expressions denote regular languages — they explicitly show the operations that were used to form the language

Notation and Parenthesis

- For a regular expression r, $L(\mathrm{r})$ is the language denoted by r. Multiple regular expressions can denote the same language!
  Example: $(0 + 1)$ and $(1 + 0)$ denote same language $\{0, 1\}$

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Some examples of regular expressions

Interpreting regular expressions

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2. $(0+1)^*001(0+1)^*$:
3. $0^* + (0^*10^*10^*10^*)^*$:
4. $(\epsilon+1)(01)^*(\epsilon+0)$:

Creating regular expressions

1. All strings that end in 1011?

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Creating regular expressions

1. All strings that end in 1011?
2. All strings except 11?
3. All strings that do not contain 000 as a subsequence?
4. All strings that do not contain the substring 10?

Tying everything together

Consider the problem of a n-input *AND* function. The input (x) is a string n-digits long with $\Sigma = \{0, 1\}$ and has an output (y) which is the logical *AND* of all the elements of x.

Formulate the regular expression which describes the above language:

Regular expressions in programming

One last expression....

Bit strings with odd number of 0s and 1s

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The regular expression is

$$(00 + 11)^*(01 + 10)$$
$$\Big(00 + 11 + (01 + 10)(00 + 11)^*(01 + 10)\Big)^*$$

Bit strings with odd number of 0s and 1s

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(Solved using techniques to be presented in the following lectures...)