Shortest Paths with Negative Lengths and DP

Lecture 18
April 1, 2021
Part I

Shortest Paths with Negative Length Edges
Single-Source Shortest Paths with Negative Edge Lengths

**Single-Source Shortest Path Problems**

**Input:** A directed graph $G = (V, E)$ with arbitrary (including negative) edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

1. Given nodes $s, t$ find shortest path from $s$ to $t$.
2. Given node $s$ find shortest path from $s$ to all other nodes.
Single-Source Shortest Paths with Negative Edge Lengths

**Single-Source Shortest Path Problems**

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1. Given nodes $s, t$ find shortest path from $s$ to $t$.
2. Given node $s$ find shortest path from $s$ to all other nodes.
What are the distances computed by Dijkstra’s algorithm?

The distance as computed by Dijkstra algorithm starting from $s$:

(A) $s = 0, x = 5, y = 1, z = 0$.

(B) $s = 0, x = 1, y = 2, z = 5$.

(C) $s = 0, x = 5, y = 1, z = 2$.

(D) IDK.
With negative length edges, Dijkstra’s algorithm can fail.
Dijkstra’s Algorithm and Negative Lengths

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![Diagram of a graph illustrating Dijkstra's algorithm with negative lengths.](image)
Dijkstra’s Algorithm and Negative Lengths

With negative length edges, Dijkstra’s algorithm can fail

Shortest path

Chandra (UIUC)
With negative length edges, Dijkstra’s algorithm can fail.

False assumption: Dijkstra’s algorithm is based on the assumption that if $s = v_0 \rightarrow v_1 \rightarrow v_2 \ldots \rightarrow v_k$ is a shortest path from $s$ to $v_k$ then $\text{dist}(s, v_i) \leq \text{dist}(s, v_{i+1})$ for $0 \leq i < k$. Holds true only for non-negative edge lengths.
Negative Length Cycles

Definition

A cycle $C$ is a negative length cycle if the sum of the edge lengths of $C$ is negative.
**Definition**

A cycle $C$ is a negative length cycle if the sum of the edge lengths of $C$ is negative.
Shortest Paths and Negative Cycles

Given $G = (V, E)$ with edge lengths and $s, t$. Suppose

1. $G$ has a negative length cycle $C$, and
2. $s$ can reach $C$ and $C$ can reach $t$.

**Question:** What is the shortest **distance** from $s$ to $t$?

Possible answers: Define shortest distance to be:

1. undefined, that is $-\infty$, OR
2. the length of a shortest **simple** path from $s$ to $t$. 
Shortest Paths and Negative Cycles

Given $G = (V, E)$ with edge lengths and $s, t$. Suppose
1. $G$ has a negative length cycle $C$, and
2. $s$ can reach $C$ and $C$ can reach $t$.

Question: What is the shortest distance from $s$ to $t$?
Possible answers: Define shortest distance to be:
1. undefined, that is $-\infty$, OR
2. the length of a shortest simple path from $s$ to $t$.

Lemma

If there is an efficient algorithm to find a shortest simple $s \rightarrow t$ path in a graph with negative edge lengths, then there is an efficient algorithm to find the longest simple $s \rightarrow t$ path in a graph with positive edge lengths.

Finding the $s \rightarrow t$ longest path is difficult. **NP-Hard!**
Given a graph $G = (V, E)$:

1. A **path** is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$.

2. A **walk** is a sequence of vertices $v_1, v_2, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k - 1$. Vertices are allowed to repeat.

Define $dist(u, v)$ to be the length of a shortest walk from $u$ to $v$.

1. If there is a walk from $u$ to $v$ that contains negative length cycle then $dist(u, v) = -\infty$

2. Else there is a path with at most $n - 1$ edges whose length is equal to the length of a shortest walk and $dist(u, v)$ is finite

Helpful to think about walks
Shortest Paths with Negative Edge Lengths

Algorithmic Problems

**Input:** A directed graph $G = (V, E)$ with edge lengths (could be negative). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

**Questions:**

1. Given nodes $s, t$, either find a negative length cycle $C$ that $s$ can reach or find a shortest path from $s$ to $t$.

2. Given node $s$, either find a negative length cycle $C$ that $s$ can reach or find shortest path distances from $s$ to all reachable nodes.

3. Check if $G$ has a negative length cycle or not.
Note: With negative lengths, shortest path problems and negative cycle detection in undirected graphs cannot be reduced to directed graphs by bi-directing each undirected edge. Why?

Problem can be solved efficiently in undirected graphs but algorithms are different and more involved than those for directed graphs. Beyond the scope of this class. If interested, ask instructor for references.
Why Negative Lengths?

Several Applications

1. Shortest path problems useful in modeling many situations — in some negative lengths are natural

2. Negative length cycle can be used to find arbitrage opportunities in currency trading

3. Important sub-routine in algorithms for more general problem: minimum-cost flow
Negative cycles
Application to Currency Trading

Currency Trading

**Input:** \( n \) currencies and for each ordered pair \((a, b)\) the *exchange rate* for converting one unit of \( a \) into one unit of \( b \).

**Questions:**

1. Is there an arbitrage opportunity?
2. Given currencies \( s, t \) what is the best way to convert \( s \) to \( t \) (perhaps via other intermediate currencies)?

Concrete example:

1. 1 Chinese Yuan = 0.1116 Euro
2. 1 Euro = 1.3617 US dollar
3. 1 US Dollar = 7.1 Chinese Yuan.

Thus, if exchanging 1 $ → Yuan → Euro → $, we get:

\[
0.1116 \times 1.3617 \times 7.1 = 1.07896$.
\]
Reducing Currency Trading to Shortest Paths

**Observation:** If we convert currency $i$ to $j$ via intermediate currencies $k_1, k_2, \ldots, k_h$ then one unit of $i$ yields $\text{exch}(i, k_1) \times \text{exch}(k_1, k_2) \times \ldots \times \text{exch}(k_h, j)$ units of $j$. 

Create currency trading directed graph $G = (V, E)$:

1. For each currency $i$ there is a node $v_i \in V$.
2. $E = V \times V$: an edge for each pair of currencies.
3. Edge length $\ell((v_i, v_j)) = \log(\text{exch}(i, j))$ can be negative.

**Exercise:**
1. There is an arbitrage opportunity if and only if $G$ has a negative length cycle.
2. The best way to convert currency $i$ to currency $j$ is via a shortest path in $G$ from $i$ to $j$. If $d$ is the distance from $i$ to $j$ then one unit of $i$ can be converted into $2^d$ units of $j$. 


Reducing Currency Trading to Shortest Paths

Observation: If we convert currency \( i \) to \( j \) via intermediate currencies \( k_1, k_2, \ldots, k_h \) then one unit of \( i \) yields 
\[
\text{exch}(i, k_1) \times \text{exch}(k_1, k_2) \ldots \times \text{exch}(k_h, j)
\] units of \( j \).

Create currency trading directed graph \( G = (V, E) \):

1. For each currency \( i \) there is a node \( v_i \in V \)
2. \( E = V \times V \): an edge for each pair of currencies
3. edge length \( \ell(v_i, v_j) = \)

Exercise:
1. There is an arbitrage opportunity if and only if \( G \) has a negative length cycle.
2. The best way to convert currency \( i \) to currency \( j \) is via a shortest path in \( G \) from \( i \) to \( j \). If \( d \) is the distance from \( i \) to \( j \) then one unit of \( i \) can be converted into \( 2^d \) units of \( j \).
Reducing Currency Trading to Shortest Paths

Observation: If we convert currency \( i \) to \( j \) via intermediate currencies \( k_1, k_2, \ldots, k_h \) then one unit of \( i \) yields \( \text{exch}(i, k_1) \times \text{exch}(k_1, k_2) \times \ldots \times \text{exch}(k_h, j) \) units of \( j \).

Create currency trading directed graph \( G = (V, E) \):

1. For each currency \( i \) there is a node \( v_i \in V \)
2. \( E = V \times V \): an edge for each pair of currencies
3. edge length \( \ell(v_i, v_j) = -\log(\text{exch}(i, j)) \) can be negative
Reducing Currency Trading to Shortest Paths

**Observation:** If we convert currency $i$ to $j$ via intermediate currencies $k_1, k_2, \ldots, k_h$ then one unit of $i$ yields $\text{exch}(i, k_1) \times \text{exch}(k_1, k_2) \ldots \times \text{exch}(k_h, j)$ units of $j$.

Create currency trading *directed* graph $G = (V, E)$:

1. For each currency $i$ there is a node $v_i \in V$
2. $E = V \times V$: an edge for each pair of currencies
3. edge length $\ell(v_i, v_j) = -\log(\text{exch}(i, j))$ can be negative

**Exercise:** Verify that

1. There is an arbitrage opportunity if and only if $G$ has a negative length cycle.
2. The best way to convert currency $i$ to currency $j$ is via a shortest path in $G$ from $i$ to $j$. If $d$ is the distance from $i$ to $j$.
1. \( \log(\alpha_1 \times \alpha_2 \times \ldots \times \alpha_k) = \log \alpha_1 + \log \alpha_2 + \ldots + \log \alpha_k. \)

2. \( \log x > 0 \) if and only if \( x > 1. \)
Lemma

Let $G$ be a directed graph with arbitrary edge lengths. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from $s$ to $v_k$ then for $1 \leq i < k$:

$s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from $s$ to $v_i$
Shortest Paths with Negative Lengths

Lemma

Let $G$ be a directed graph with arbitrary edge lengths. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from $s$ to $v_k$ then for $1 \leq i < k$:

1. $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from $s$ to $v_i$
2. False: $\text{dist}(s, v_i) \leq \text{dist}(s, v_k)$ for $1 \leq i < k$. Holds true only for non-negative edge lengths.
Lemma

Let $G$ be a directed graph with arbitrary edge lengths. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from $s$ to $v_k$ then for $1 \leq i < k$:

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2. False: $\text{dist}(s, v_i) \leq \text{dist}(s, v_k)$ for $1 \leq i < k$. Holds true only for non-negative edge lengths.

Cannot explore nodes in increasing order of distance! We need other strategies.
Shortest Paths and Recursion

1. Compute the shortest path distance from $s$ to $t$ recursively?
2. What are the smaller sub-problems?

Lemma
Let $G$ be a directed graph with arbitrary edge lengths. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from $s$ to $v_k$ then for $1 \leq i < k$: $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from $s$ to $v_i$.

Sub-problem idea: paths of fewer hops/edges.
1. Compute the shortest path distance from $s$ to $t$ recursively?
2. What are the smaller sub-problems?

**Lemma**

Let $G$ be a directed graph with arbitrary edge lengths. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from $s$ to $v_k$ then for $1 \leq i < k$:

1. $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from $s$ to $v_i$.
1. Compute the shortest path distance from \( s \) to \( t \) recursively?
2. What are the smaller sub-problems?

**Lemma**

Let \( G \) be a directed graph with arbitrary edge lengths. If \( s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k \) is a shortest path from \( s \) to \( v_k \) then for \( 1 \leq i < k \):

1. \( s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i \) is a shortest path from \( s \) to \( v_i \)

Sub-problem idea: paths of fewer hops/edges
Hop-based Recursion: Bellman-Ford Algorithm

Single-source problem: fix source $s$.
Assume that all nodes can be reached by $s$ in $G$
Assume $G$ has no negative-length cycle (for now).

$d(v, k)$: shortest walk length from $s$ to $v$ using at most $k$ edges.

$d(s, 0) = 0$
$d(s, 0) = \infty$ for $v \neq s$
Hop-based Recursion: Bellman-Ford Algorithm

Single-source problem: fix source $s$. Assume that all nodes can be reached by $s$ in $G$. Assume $G$ has no negative-length cycle (for now).

$d(v, k)$: shortest walk length from $s$ to $v$ using at most $k$ edges. Note: $\text{dist}(s, v) = d(v, n - 1)$.

$d(s, 1) = 0$
$d(v, 1) = \infty$ if $(s, v)$ edge
otherwise $d(s, v)$. 

$\text{dist}(s, v) = d(v, n - 1)$. 

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Note: $\text{dist}(s, v) = d(v, n - 1)$. Recursion for $d(v, k)$:
Hop-based Recursion: Bellman-Ford Algorithm

Single-source problem: fix source $s$.
Assume that all nodes can be reached by $s$ in $G$
Assume $G$ has no negative-length cycle (for now).

$d(v, k)$: shortest walk length from $s$ to $v$ using at most $k$ edges.
Note: $\text{dist}(s, v) = d(v, n - 1)$.

Recursion for $d(v, k)$:

$$d(v, k) = \min \left\{ \min_{u \in V} (d(u, k - 1) + \ell(u, v)), d(v, k - 1) \right\}.$$

Base case: $d(s, 0) = 0$ and $d(v, 0) = \infty$ for all $v \neq s$. 
$$d(u, d(v, k) = \min \left\{ \begin{array}{l} d(u_1, k-1) + l(u, v) \\
 d(u_2, k-1) + l(u_2, v) \\
 d(u_3, k-1) + l(u_3, k) \\
 d(v, k-1). \end{array} \right. \right.$$ 

$$d(v, 1) = \min \left\{ \begin{array}{l} \min_{u \in \text{In}(v)} d(u, v) + l(u, v) \\
 d(v, 0) \end{array} \right. \right.$$
Example

\[ d(b, z) = \min_y \{ d(b, y) \} \]

\[ d(c, z) + l(c, b) \]

\[ d(f, z) \]
Bellman-Ford Algorithm

\begin{center}
\begin{verbatim}
for each $u \in V$ do \\
    $d(u, 0) \leftarrow \infty$ \\
$d(s, 0) \leftarrow 0$

for $k = 1$ to $n - 1$ do \\
    for each $v \in V$ do \\
        $d(v, k) \leftarrow d(v, k - 1)$ \\
        for each edge $(u, v) \in \text{ln}(v)$ do \\
            $d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\}$

for each $v \in V$ do \\
    $\text{dist}(s, v) \leftarrow d(v, n - 1)$
\end{verbatim}
\end{center}
Bellman-Ford Algorithm

for each $u \in V$ do
  $d(u, 0) \leftarrow \infty$
  $d(s, 0) \leftarrow 0$

for $k = 1$ to $n - 1$ do
  for each $v \in V$ do
    $d(v, k) \leftarrow d(v, k - 1)$
    for each edge $(u, v) \in \text{In}(v)$ do
      $d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\}$

for each $v \in V$ do
  $\text{dist}(s, v) \leftarrow d(v, n - 1)$

Running time:
Bellman-Ford Algorithm

for each $u \in V$ do
    $d(u, 0) \leftarrow \infty$
    $d(s, 0) \leftarrow 0$

for $k = 1$ to $n - 1$ do
    for each $v \in V$ do
        $d(v, k) \leftarrow d(v, k - 1)$
    for each edge $(u, v) \in \text{ln}(v)$ do
        $d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\}$

for each $v \in V$ do
    dist$(s, v) \leftarrow d(v, n - 1)$

Running time: $O(mn)$
Bellman-Ford Algorithm

for each $u \in V$ do
    $d(u, 0) \leftarrow \infty$
    $d(s, 0) \leftarrow 0$

for $k = 1$ to $n - 1$ do
    for each $v \in V$ do
        for each $u \in \text{ln}(v)$ do
            $d(v, k) \leftarrow d(v, k - 1)$
        $d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\}$

for each $v \in V$ do
    $\text{dist}(s, v) \leftarrow d(v, n - 1)$

Running time: $O(mn)$
Space:
Bellman-Ford Algorithm

for each $u \in V$ do
  $d(u, 0) \leftarrow \infty$
  $d(s, 0) \leftarrow 0$

for $k = 1$ to $n - 1$ do
  for each $v \in V$ do
    $d(v, k) \leftarrow d(v, k - 1)$
    for each edge $(u, v) \in \text{ln}(v)$ do
      $d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\}$
  
for each $v \in V$ do
  $\text{dist}(s, v) \leftarrow d(v, n - 1)$

Running time: $O(mn)$ Space: $O(m + n^2)$
Bellman-Ford Algorithm

\begin{algorithm}
\begin{algorithmic}
  \State \textbf{for each} $u \in V$ \textbf{do}
  \State \hspace{1em} $d(u, 0) \leftarrow \infty$
  \State \hspace{1em} $d(s, 0) \leftarrow 0$
  \State \textbf{for} $k = 1$ \textbf{to} $n - 1$ \textbf{do}
  \State \hspace{1em} \textbf{for each} $v \in V$ \textbf{do}
  \State \hspace{2em} $d(v, k) \leftarrow d(v, k - 1)$
  \State \hspace{2em} \textbf{for each edge} $(u, v) \in \text{ln}(v)$ \textbf{do}
  \State \hspace{3em} $d(v, k) = \min\{d(v, k), d(u, k - 1) + \ell(u, v)\}$
  \State \textbf{for each} $v \in V$ \textbf{do}
  \State \hspace{1em} $\text{dist}(s, v) \leftarrow d(v, n - 1)$
\end{algorithmic}
\end{algorithm}

Running time: $O(mn)$ Space: $O(m + n^2)$
Space can be reduced to $O(m + n)$. 
Bellman-Ford Algorithm

for each $u \in V$ do
    $d(u) \leftarrow \infty$
    $d(s) \leftarrow 0$

for $k = 1$ to $n - 1$ do
    for each $v \in V$ do
        for each edge $(u, v) \in \text{ln}(v)$ do
            $d(v) = \min\{d(v), d(u) + \ell(u, v)\}$

for each $v \in V$ do
    $\text{dist}(s, v) \leftarrow d(v)$

Running time: $O(mn)$ Space: $O(m + n)$

Exercise: Argue that this achieves same results as algorithm on previous slide.
Check if distances change in iteration $n$.

\begin{verbatim}
for each $u \in V$ do
    $d(u) \leftarrow \infty$
    $d(s) \leftarrow 0$

for $k = 1$ to $n - 1$ do
    for each $v \in V$ do
        for each edge $(u, v) \in ln(v)$ do
            $d(v) = \min\{d(v), d(u) + \ell(u, v)\}$

(* One more iteration to check if distances change *)

for each $v \in V$ do
    for each edge $(u, v) \in ln(v)$ do
        if $(d(v) > d(u) + \ell(u, v))$
            Output ‘‘Negative Cycle’’

for each $v \in V$ do
    $\text{dist}(s, v) \leftarrow d(v)$
\end{verbatim}
Correctness of the Bellman-Ford Algorithm

Via induction: For each $v$, $d(v, k)$ is the length of a shortest walk from $s$ to $v$ with at most $k$ hops.
Correctness of the Bellman-Ford Algorithm

Via induction: For each $v$, $d(v, k)$ is the length of a shortest walk from $s$ to $v$ with at most $k$ hops.

Lemma

Suppose $G$ does not have a negative length cycle reachable from $s$. Then for all $v$, $\text{dist}(s, v) = d(v, n - 1)$. Moreover, $d(v, n - 1) = d(v, n)$. 

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Correctness of the Bellman-Ford Algorithm

Via induction: For each $v$, $d(v, k)$ is the length of a shortest walk from $s$ to $v$ with at most $k$ hops.

Lemma

Suppose $G$ does not have a negative length cycle reachable from $s$. Then for all $v$, $\text{dist}(s, v) = d(v, n - 1)$. Moreover, $d(v, n - 1) = d(v, n)$.

Proof.

Consider a shortest $s$-$v$ walk. If it is a path then it has at most $n - 1$ edges and we are done. If not it has a cycle and we can remove it (since cycle length is non-negative) from walk to obtain a shorter walk or a shortest walk with fewer edges.

Corollary: Bellman-Ford correctly outputs the shortest path distances if $G$ has no negative length cycle reachable from $s$. 
Correctness: detecting negative length cycle

**Lemma**

\[ G \text{ has a negative length cycle reachable from } s \text{ if and only if there is some node } v \text{ such that } d(v, n) < d(v, n - 1). \]

Lemma proves correctness of negative cycle detection by Bellman-Ford algorithm.

The only if direction follows from Lemma on previous slide. We prove the if direction in the next slide.
Suppose $G$ has a negative cycle $C$ reachable from $s$. Then there is some node $v \in C$ such that $d(v, n) < d(v, n - 1)$. 
Correctness: detecting negative length cycle

Lemma

Suppose $G$ has a negative cycle $C$ reachable from $s$. Then there is some node $v \in C$ such that $d(v, n) < d(v, n - 1)$.

Proof.

Suppose not. Let $C = v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_h \rightarrow v_1$ be negative length cycle reachable from $s$. $d(v_i, n - 1)$ is finite for $1 \leq i \leq h$ since $C$ is reachable from $s$. By assumption $d(v, n) \geq d(v, n - 1)$ for all $v \in C$; implies no change in $n$'th iteration;

$d(v_i, n - 1) = d(v_i, n)$ for $1 \leq i \leq h$. This means

$d(v_i, n - 1) \leq d(v_{i-1}, n - 1) + \ell(v_{i-1}, v_i)$ for $2 \leq i \leq h$ and

$d(v_1, n - 1) \leq d(v_n, n - 1) + \ell(v_n, v_1)$. Adding up all these inequalities results in the inequality $0 \leq \ell(C)$ which contradicts the assumption that $\ell(C) < 0$. 

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CS/ECE 374  
Spring 2021
Finding the Paths and a Shortest Path Tree

How do we find a shortest path tree in addition to distances?

- For each $v$ the $d(v)$ can only get smaller as algorithm proceeds.
- If $d(v)$ becomes smaller it is because we found a vertex $u$ such that $d(v) > d(u) + \ell(u, v)$ and we update $d(v) = d(u) + \ell(u, v)$. That is, we found a shorter path to $v$ through $u$.
- For each $v$ have a $\text{prev}(v)$ pointer and update it to point to $u$ if $v$ finds a shorter path via $u$.
- At end of algorithm $\text{prev}(v)$ pointers give a shortest path tree oriented towards the source $s$. 
Negative Cycle Detection

Given directed graph $G$ with arbitrary edge lengths, does it have a negative length cycle?

Exercise: How will you actually find a negative length cycle if it exists?
Negative Cycle Detection

Given directed graph $G$ with arbitrary edge lengths, does it have a negative length cycle?

**Exercise:** How will you actually find a negative length cycle if it exists?

1. Bellman-Ford checks whether there is a negative cycle $C$ that is reachable from a specific vertex $s$. There may be negative cycles not reachable from $s$.
2. Run Bellman-Ford $|V|$ times, once from each node $u$?
Negative Cycle Detection

1. Add a new node $s'$ and connect it to all nodes of $G$ with zero length edges. Bellman-Ford from $s'$ will fill find a negative length cycle if there is one. **Exercise:** why does this work?

2. Negative cycle detection can be done with one Bellman-Ford invocation.
Part II

Shortest Paths in DAGs
Shortest Paths in a DAG

Single-Source Shortest Path Problems

**Input** A directed acyclic graph $G = (V, E)$ with arbitrary (including negative) edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

1. Given nodes $s, t$ find shortest path from $s$ to $t$.
2. Given node $s$ find shortest path from $s$ to all other nodes.
Shortest Paths in a **DAG**

**Single-Source Shortest Path Problems**

**Input** A directed acyclic graph $G = (V, E)$ with arbitrary (including negative) edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

1. Given nodes $s, t$ find shortest path from $s$ to $t$.
2. Given node $s$ find shortest path from $s$ to all other nodes.

Simplification of algorithms for **DAGs**

1. No cycles and hence no negative length cycles! Hence can find shortest paths even for negative length edges
2. Can order nodes using topological sort
Algorithm for **DAGs**

1. Want to find shortest paths from $s$. Ignore nodes not reachable from $s$.
2. Let $s = v_1, v_2, v_{i+1}, \ldots, v_n$ be a topological sort of $G$. 

Observation:
1. shortest path from $s$ to $v_i$ cannot use any node from $v_{i+1}, \ldots, v_n$.
2. can find shortest paths in topological sort order.
Algorithm for **DAGs**

1. Want to find shortest paths from $s$. Ignore nodes not reachable from $s$.
2. Let $s = v_1, v_2, v_{i+1}, \ldots, v_n$ be a topological sort of $G$

**Observation:**

1. shortest path from $s$ to $v_i$ cannot use any node from $v_{i+1}, \ldots, v_n$
2. can find shortest paths in topological sort order.
Algorithm for **DAGs**

Assumption: \( s \) is first in the topological sort

```plaintext
for \( i = 1 \) to \( n \) do 
    \( d(s, v_i) = \infty \) 
    \( d(s, s) = 0 \) 

for \( i = 1 \) to \( n - 1 \) do 
    for each edge \((v_i, v_j)\) in \( \text{Adj}(v_i) \) do 
        \( d(s, v_j) = \min\{d(s, v_j), d(s, v_i) + \ell(v_i, v_j)\} \)

return \( d(s, \cdot) \) values computed
```

**Correctness:** induction on \( i \) and observation in previous slide.

**Running time:** \( O(m + n) \) time algorithm! Works for negative edge lengths and hence can find *longest* paths in a **DAG**.
Algorithm for **DAGs**, a variant

Assumption: $s$ is first in the topological sort

```
for $i = 1$ to $n$ do
  $d(s, v_i) = \infty$
  $d(s, s) = 0$

for $i = 2$ to $n - 1$ do
  for each edge $(v_j, v_i)$ in $\text{In}(v_i)$ do
    $d(s, v_i) = \min\{d(s, v_i), d(s, v_j) + \ell(v_j, v_i)\}$

return $d(s, \cdot)$ values computed
```

When visiting $v_i$ scan incoming edges to find shortest path to $i$. Previous algorithm scanned all edges in $\text{Adj}(v_i)$ after processing $v_i$. Can see algorithms are same.
Algorithm for **DAGs**: Example

Want distances from \textit{a} say. Consider topological sort: \textit{a, b, c, e, g, d, f, h}
Bellman-Ford and DAGs

Bellman-Ford is based on the following principles:
- The shortest walk length from $s$ to $v$ with at most $k$ hops can be computed via dynamic programming.
- $G$ has a negative length cycle reachable from $s$ iff there is a node $v$ such that shortest walk length reduces after $n$ hops.

We can find hop-constrained shortest paths via graph reduction. Given $G = (V, E)$ with edge lengths $\ell(e)$ and integer $k$ construction new layered graph $G' = (V', E')$ as follows:
- $V' = V \times \{0, 1, 2, \ldots, k\}$.
- $E' = \{(((u, i), (v, i + 1) | (u, v) \in E, 0 \leq i < k\}$,
  $\ell(((u, i), (v, i + 1)) = \ell(u, v)$

**Lemma**

*Shortest path distance from $(u, 0)$ to $(v, k)$ in $G'$ is equal to the shortest walk from $u$ to $v$ in $G$ with exactly $k$ edges.*
Layered DAG: Figure
Part III

All Pairs Shortest Paths
Shortest Path Problems

**Input**  A (undirected or directed) graph $G = (V, E)$ with edge lengths (or costs). For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

1. Given nodes $s, t$ find shortest path from $s$ to $t$.
2. Given node $s$ find shortest path from $s$ to all other nodes.
3. Find shortest paths for all pairs of nodes.
### Single-Source Shortest Path Problems

**Input** A (undirected or directed) graph $G = (V, E)$ with edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

1. Given nodes $s, t$ find shortest path from $s$ to $t$.
2. Given node $s$ find shortest path from $s$ to all other nodes.
# Single-Source Shortest Paths

## Single-Source Shortest Path Problems

**Input** A (undirected or directed) graph $G = (V, E)$ with edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

1. Given nodes $s, t$ find shortest path from $s$ to $t$.
2. Given node $s$ find shortest path from $s$ to all other nodes.

**Dijkstra’s algorithm** for non-negative edge lengths. Running time: $O((m + n) \log n)$ with heaps and $O(m + n \log n)$ with advanced priority queues.

**Bellman-Ford algorithm** for arbitrary edge lengths. Running time: $O(nm)$. 
All-Pairs Shortest Paths

**All-Pairs Shortest Path Problem**

**Input** A (undirected or directed) graph $G = (V, E)$ with edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

1. Find shortest paths for all pairs of nodes.
All-Pairs Shortest Paths

**All-Pairs Shortest Path Problem**

**Input** A (undirected or directed) graph \( G = (V, E) \) with edge lengths. For edge \( e = (u, v) \), \( \ell(e) = \ell(u, v) \) is its length.

1. Find shortest paths for all pairs of nodes.

Apply single-source algorithms \( n \) times, once for each vertex.

1. Non-negative lengths. \( O(nm \log n) \) with heaps and \( O(nm + n^2 \log n) \) using advanced priority queues.

2. Arbitrary edge lengths: \( O(n^2m) \).
\( \Theta(n^4) \) if \( m = \Omega(n^2) \).
# All-Pairs Shortest Paths

## All-Pairs Shortest Path Problem

**Input**  A (undirected or directed) graph $G = (V, E)$ with edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

1. Find shortest paths for all pairs of nodes.

Apply single-source algorithms $n$ times, once for each vertex.

1. Non-negative lengths. $O(nm \log n)$ with heaps and $O(nm + n^2 \log n)$ using advanced priority queues.

2. Arbitrary edge lengths: $O(n^2 m)$.  
   $\Theta(n^4)$ if $m = \Omega(n^2)$.

Can we do better?
All-Pairs: Recursion on index of intermediate nodes

1. Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$

2. $\text{dist}(i, j, k)$: length of shortest walk from $v_i$ to $v_j$ among all walks in which the largest index of an intermediate node is at most $k$ (could be $-\infty$ if there is a negative length cycle).

```
\begin{align*}
\text{dist}(i, j, 0) & = \\
\text{dist}(i, j, 1) & = \\
\text{dist}(i, j, 2) & = \\
\text{dist}(i, j, 3) & = 
\end{align*}
```
All-Pairs: Recursion on index of intermediate nodes

1. Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$

2. $\text{dist}(i, j, k)$: length of shortest walk from $v_i$ to $v_j$ among all walks in which the largest index of an intermediate node is at most $k$ (could be $-\infty$ if there is a negative length cycle).

$$\begin{align*}
\text{dist}(i, j, 0) &= 100 \\
\text{dist}(i, j, 1) &= \\
\text{dist}(i, j, 2) &= \\
\text{dist}(i, j, 3) &=
\end{align*}$$

![Graph diagram with labels and distances]
All-Pairs: Recursion on index of intermediate nodes

1. Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$

2. $\text{dist}(i, j, k)$: length of shortest walk from $v_i$ to $v_j$ among all walks in which the largest index of an intermediate node is at most $k$ (could be $-\infty$ if there is a negative length cycle).

![Graph](image)

$\text{dist}(i, j, 0) = 100$
$\text{dist}(i, j, 1) = 9$
$\text{dist}(i, j, 2) =$
$\text{dist}(i, j, 3) =$
All-Pairs: Recursion on index of intermediate nodes

1. Number vertices arbitrarily as \( v_1, v_2, \ldots, v_n \)

2. \( \text{dist}(i, j, k) \): length of shortest walk from \( v_i \) to \( v_j \) among all walks in which the largest index of an intermediate node is at most \( k \) (could be \(-\infty\) if there is a negative length cycle).

\[ 
\begin{align*}
\text{dist}(i, j, 0) &= 100 \\
\text{dist}(i, j, 1) &= 9 \\
\text{dist}(i, j, 2) &= 8 \\
\text{dist}(i, j, 3) &= \\
\end{align*}
\]
All-Pairs: Recursion on index of intermediate nodes

1. Number vertices arbitrarily as $v_1, v_2, \ldots, v_n$

2. $\text{dist}(i, j, k)$: length of shortest walk from $v_i$ to $v_j$ among all walks in which the largest index of an intermediate node is at most $k$ (could be $-\infty$ if there is a negative length cycle).

$\text{dist}(i, j, 0) = 100$
$\text{dist}(i, j, 1) = 9$
$\text{dist}(i, j, 2) = 8$
$\text{dist}(i, j, 3) = 5$
For the following graph, $\text{dist}(i, j, 2)$ is...
All-Pairs: Recursion on index of intermediate nodes

\[ \text{dist}(i, k, k - 1) \rightarrow k \rightarrow \text{dist}(k, j, k - 1) \]

\[ \text{dist}(i, j, k - 1) \]

\[ \text{dist}(i, j, k) = \min \left\{ \begin{array}{l}
\text{dist}(i, j, k - 1) \\
\text{dist}(i, k, k - 1) + \text{dist}(k, j, k - 1)
\end{array} \right\} \]

Base case: \( \text{dist}(i, j, 0) = \ell(i, j) \) if \((i, j) \in E\), otherwise \( \infty \)

Correctness: If \( i \to j \) shortest walk goes through \( k \) then \( k \) occurs only once on the path — otherwise there is a negative length cycle.
All-Pairs: Recursion on index of intermediate nodes

If $i$ can reach $k$ and $k$ can reach $j$ and $\text{dist}(k, k, k - 1) < 0$ then $G$ has a negative length cycle containing $k$ and $\text{dist}(i, j, k) = -\infty$.

Recursion below is valid only if $\text{dist}(k, k, k - 1) \geq 0$. We can detect this during the algorithm or wait till the end.

$$\text{dist}(i, j, k) = \min \left\{ \text{dist}(i, j, k - 1), \text{dist}(i, k, k - 1) + \text{dist}(k, j, k - 1) \right\}$$
Floyd-Warshall Algorithm
for All-Pairs Shortest Paths

\[
\begin{align*}
\text{for } i = 1 \text{ to } n & \text{ do} \\
& \hspace{1em} \text{for } j = 1 \text{ to } n \text{ do} \\
& \hspace{2em} \text{dist}(i, j, 0) = \ell(i, j) \quad (\ast \quad \ell(i, j) = \infty \text{ if } (i, j) \notin E, \ 0 \text{ if } i = j \quad \ast) \\
\text{for } k = 1 \text{ to } n & \text{ do} \\
& \hspace{1em} \text{for } i = 1 \text{ to } n \text{ do} \\
& \hspace{2em} \text{for } j = 1 \text{ to } n \text{ do} \\
& \hspace{3em} \text{dist}(i, j, k) = \min \left\{ \text{dist}(i, j, k - 1), \right. \\
& \hspace{5em} \left. \text{dist}(i, k, k - 1) + \text{dist}(k, j, k - 1) \right\} \\
\text{for } i = 1 \text{ to } n & \text{ do} \\
& \hspace{1em} \text{if } (\text{dist}(i, i, n) < 0) \text{ then} \\
& \hspace{2em} \text{Output that there is a negative length cycle in } G
\end{align*}
\]
Floyd-Warshall Algorithm
for All-Pairs Shortest Paths

for $i = 1$ to $n$ do
  for $j = 1$ to $n$ do
    $dist(i, j, 0) = \ell(i, j)$ (* $\ell(i, j) = \infty$ if $(i, j) \notin E$, 0 if $i = j$ *)

for $k = 1$ to $n$ do
  for $i = 1$ to $n$ do
    for $j = 1$ to $n$ do
      $dist(i, j, k) = \min\left\{\begin{array}{l}
        dist(i, j, k - 1), \\
        dist(i, k, k - 1) + dist(k, j, k - 1)
      \end{array}\right.$

for $i = 1$ to $n$ do
  if ($dist(i, i, n) < 0$) then
    Output that there is a negative length cycle in $G$

Running Time:
Floyd-Warshall Algorithm
for All-Pairs Shortest Paths

for \( i = 1 \) to \( n \) do
  for \( j = 1 \) to \( n \) do
    \[ \text{dist}(i, j, 0) = \ell(i, j) \] (* \( \ell(i, j) = \infty \) if \( (i, j) \notin E \), 0 if \( i = j \) *)

for \( k = 1 \) to \( n \) do
  for \( i = 1 \) to \( n \) do
    for \( j = 1 \) to \( n \) do
      \[ \text{dist}(i, j, k) = \min \left\{ \text{dist}(i, j, k - 1), \text{dist}(i, k, k - 1) + \text{dist}(k, j, k - 1) \right\} \]

for \( i = 1 \) to \( n \) do
  if \( \text{dist}(i, i, n) < 0 \) then
    Output that there is a negative length cycle in \( G \)

Running Time: \( \Theta(n^3) \), Space: \( \Theta(n^3) \).
Floyd-Warshall Algorithm
for All-Pairs Shortest Paths

for $i = 1$ to $n$ do
  for $j = 1$ to $n$ do
    $dist(i, j, 0) = \ell(i, j)$ (* $\ell(i, j) = \infty$ if $(i, j) \notin E$, 0 if $i = j$ *)

for $k = 1$ to $n$ do
  for $i = 1$ to $n$ do
    for $j = 1$ to $n$ do
      $dist(i, j, k) = \min\left\{ dist(i, j, k - 1),
      \begin{cases} dist(i, k, k - 1), \\
      dist(i, k, k - 1) + dist(k, j, k - 1) \end{cases} \right\}$

for $i = 1$ to $n$ do
  if ($dist(i, i, n) < 0$) then
    Output that there is a negative length cycle in $G$

Running Time: $\Theta(n^3)$, Space: $\Theta(n^3)$.
Correctness: via induction and recursive definition
Question: Can we find the paths in addition to the distances?
Floyd-Warshall Algorithm: Finding the Paths

Question: Can we find the paths in addition to the distances?

1. Create a $n \times n$ array `Next` that stores the next vertex on shortest path for each pair of vertices.
2. With array `Next`, for any pair of given vertices $i, j$ can compute a shortest path in $O(n)$ time.
Floyd-Warshall Algorithm
Finding the Paths

for $i = 1$ to $n$ do
  for $j = 1$ to $n$ do
    \( \text{dist}(i, j, 0) = \ell(i, j) \)
  
  (* \( \ell(i, j) = \infty \) if \((i, j)\) not edge, 0 if \(i = j\) *)
  \( \text{Next}(i, j) = -1 \)

for $k = 1$ to $n$ do
  for $i = 1$ to $n$ do
    for $j = 1$ to $n$ do
      if (dist\(i, j, k - 1\) > dist\(i, k, k - 1\) + dist\(k, j, k - 1\)) then
        dist\(i, j, k\) = dist\(i, k, k - 1\) + dist\(k, j, k - 1\)
        Next\(i, j\) = $k$

for $i = 1$ to $n$ do
  if (dist\(i, i, n\) < 0) then
    Output that there is a negative length cycle in \(G\)

Exercise: Given \textbf{Next} array and any two vertices \(i, j\) describe an \(O(n)\) algorithm to find a \(i-j\) shortest path.
# Summary of results on shortest paths

<table>
<thead>
<tr>
<th>Single source</th>
<th>Dijkstra</th>
<th>Bellman Ford</th>
</tr>
</thead>
<tbody>
<tr>
<td>No negative edges</td>
<td>$O(n \log n + m)$</td>
<td>$O(nm)$</td>
</tr>
<tr>
<td>Edge lengths can be negative</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

## All Pairs Shortest Paths

<table>
<thead>
<tr>
<th>No negative edges</th>
<th>$n$ * Dijkstra</th>
<th>$O(n^2 \log n + nm)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No negative cycles</td>
<td>$n$ * Bellman Ford</td>
<td>$O(n^2 m) = O(n^4)$</td>
</tr>
<tr>
<td>No negative cycles</td>
<td>BF + $n$ * Dijkstra</td>
<td>$O(nm + n^2 \log n)$</td>
</tr>
<tr>
<td>No negative cycles</td>
<td>Floyd-Warshall</td>
<td>$O(n^3)$</td>
</tr>
<tr>
<td>Unweighted</td>
<td>Matrix multiplication</td>
<td>$O(n^{2.38}), O(n^{2.58})$</td>
</tr>
</tbody>
</table>
Dynamic Programming: Postscript

Dynamic Programming = Smart Recursion + Memoization

How to come up with the recursion?

How to recognize that dynamic programming may apply?
Dynamic Programming = Smart Recursion + Memoization

1. How to come up with the recursion?
2. How to recognize that dynamic programming may apply?
Some Tips

1. Problems where there is a *natural* linear ordering: sequences, paths, intervals, **DAGs** etc. Recursion based on ordering (left to right or right to left or topological sort) usually works.

2. Problems involving trees: recursion based on subtrees.

3. More generally:
   1. Problem admits a natural recursive divide and conquer
   2. If optimal solution for whole problem can be simply composed from optimal solution for each separate pieces then plain divide and conquer works directly
   3. If optimal solution depends on all pieces then can apply dynamic programming if *interface/interaction* between pieces is *limited*. Augment recursion to not simply find an optimum solution but also an optimum solution for each possible way to interact with the other pieces.
Examples

1. Longest Increasing Subsequence: break sequence in the middle say. What is the interaction between the two pieces in a solution?

2. Sequence Alignment: break both sequences in two pieces each. What is the interaction between the two sets of pieces?

3. Independent Set in a Tree: break tree at root into subtrees. What is the interaction between the subtrees?

4. Independent Set in a graph: break graph into two graphs. What is the interaction? Very high!

5. Knapsack: Split items into two sets of half each. What is the interaction?