Directed Graphs and DFS

Lecture 16
March 23, 2021
Topics

- Structure of directed graphs
  - Directed acyclic graphs (DAGs) and topological sort
  - Strong connected components and meta graph representation
- **DFS** and its properties including pre/post numbering
- Linear time algorithm for computing all SCCs of a directed graph
Strong Connected Components (SCCs)

**Algorithmic Problem**

Find all SCCs of a given directed graph.

Previous lecture:
Saw an $O(n \cdot (n + m))$ time algorithm.
This lecture: sketch of a $O(n + m)$ time algorithm.
Graph of SCCs

Meta-graph of SCCs

Let $S_1, S_2, \ldots, S_k$ be the strong connected components (i.e., SCCs) of $G$. The graph of SCCs is $G^{SCC}$

1. Vertices are $S_1, S_2, \ldots, S_k$

2. There is an edge $(S_i, S_j)$ if there is some $u \in S_i$ and $v \in S_j$ such that $(u, v)$ is an edge in $G$. 
Reversal and SCCs

Proposition

For any graph $G$, the graph of SCCs of $G^{\text{rev}}$ is the same as the reversal of $G^{\text{SCC}}$.

Proof.

Exercise.
**Proposition**

For any graph \( G \), the graph \( G^{\text{SCC}} \) has no directed cycle.

**Proof.**

If \( G^{\text{SCC}} \) has a cycle \( S_1, S_2, \ldots, S_k \) then \( S_1 \cup S_2 \cup \cdots \cup S_k \) should be in the same SCC in \( G \). Formal details: exercise.
Part I

Directed Acyclic Graphs
Directed Acyclic Graphs

**Definition**

A directed graph $G$ is a **directed acyclic graph** (DAG) if there is no directed cycle in $G$. 

![DAG Diagram]

1. Directed graph $G$ with vertices 1, 2, 3, and 4 connected in a way that forms a directed acyclic graph (DAG).
DAGs can be complicated
DAGs can be complicated
DAGs can be dense

A DAG on $n$ vertices can have $\Omega(n^2)$ edges. Which ones?
**Sources and Sinks**

**Definition**

1. A vertex $u$ is a **source** if it has no in-coming edges.
2. A vertex $u$ is a **sink** if it has no out-going edges.
DAG Properties

Proposition
Every DAG $G$ has at least one source and at least one sink.

Proof.
Let $P = v_1, v_2, \ldots, v_k$ be a longest (or a maximal) path in $G$.

Claim that $v_1$ is a source and $v_k$ is a sink. Suppose $v_1$ is not a source. Then $v_1$ has an incoming edge which either creates a cycle, or creates a longer path, both of which are contradictions. Similarly if $v_k$ is not a sink, then it has an outgoing edge and creates a cycle, or a longer path.
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Claim that \( v_1 \) is a source and \( v_k \) is a sink. Suppose \( v_1 \) is not a source. Then \( v_1 \) has an incoming edge which either creates a cycle, or creates a longer path, both of which are contradictions. Similarly if \( v_k \) is not a sink, then it has an outgoing edge and creates a cycle, or a longer path.

1. G is a DAG if and only if \( G^{\text{rev}} \) is a DAG.
2. G is a DAG if and only if each node is in its own strong connected component.

Formal proofs: exercise.
Definition

A **topological ordering/topological sorting** of $G = (V, E)$ is an ordering $\prec$ on $V$ such that if $(u \to v) \in E$ then $u \prec v$.

Informal equivalent definition:

One can order the vertices of the graph along a line (say the $x$-axis) such that all edges are from left to right.
Lemma

A directed graph $G$ can be topologically ordered if and only if it is a DAG.

Need to show both directions.
Lemma

A directed graph $G$ can be topologically ordered if it is a DAG.

Proof.
Consider the following algorithm:

1. Pick a source $u$, output it.
2. Remove $u$ and all edges out of $u$.
3. Repeat until graph is empty.

Exercise: prove this gives topological sort.

Exercise: show algorithm can be implemented in $O(m + n)$ time.
Topological Sort: Example

a → b → c

d → e → f → h → g
Lemma

A directed graph $G$ can be topologically ordered only if it is a DAG.

Proof.

Suppose $G$ is not a DAG and has a topological ordering $\prec$. $G$ has a cycle $C = u_1, u_2, \ldots, u_k, u_1$. Then $u_1 \prec u_2 \prec \ldots \prec u_k \prec u_1$!

That is... $u_1 \prec u_1$.

A contradiction (to $\prec$ being an order).

Not possible to topologically order the vertices.
**DAGs and Topological Sort**

**Note:** A **DAG** $G$ may have many different topological sorts.

**Question:** What is a **DAG** with the most number of distinct topological sorts for a given number $n$ of vertices?

**Question:** What is a **DAG** with the least number of distinct topological sorts for a given number $n$ of vertices?
Cycles in graphs

**Question:** Given an *undirected* graph how do we check whether it has a cycle and output one if it has one?

**Question:** Given an *directed* graph how do we check whether it has a cycle and output one if it has one?
To Remember: Structure of Graphs

**Undirected graph**: connected components of $G = (V, E)$ partition $V$ and can be computed in $O(m + n)$ time.

**Directed graph**: the meta-graph $G^{SCC}$ of $G$ can be computed in $O(m + n)$ time. $G^{SCC}$ gives information on the partition of $V$ into strong connected components and how they form a DAG structure.

Above structural decomposition will be useful in several algorithms.
Part II

Depth First Search (DFS)
Depth First Search

**DFS** is a special case of Basic Search but is a versatile graph exploration strategy. John Hopcroft and Bob Tarjan (Turing Award winners) demonstrated the power of **DFS** to understand graph structure. **DFS** can be used to obtain linear time \((O(m + n))\) algorithms for

1. Finding cut-edges and cut-vertices of undirected graphs
2. Finding strong connected components of directed graphs
3. Linear time algorithm for testing whether a graph is planar

Many other applications as well.
**DFS in Undirected Graphs**

Recursive version. Easier to understand some properties.

\[
\text{DFS}(G)
\]

\[
\text{for all } u \in V(G) \text{ do}
\]

\[
\begin{align*}
\text{Mark } u \text{ as unvisited} \\
\text{Set } \text{pred}(u) \text{ to null} \\
T \text{ is set to } \emptyset \\
\text{while } \exists \text{ unvisited } u \text{ do}
\end{align*}
\]

\[
\begin{align*}
\text{DFS}(u) \\
\text{Output } T
\end{align*}
\]

\[
\text{DFS}(u)
\]

\[
\begin{align*}
\text{Mark } u \text{ as visited} \\
\text{for each } uv \text{ in } \text{Out}(u) \text{ do}
\end{align*}
\]

\[
\begin{align*}
\text{if } v \text{ is not visited then}
\end{align*}
\]

\[
\begin{align*}
\text{add edge } uv \text{ to } T \\
\text{set } \text{pred}(v) \text{ to } u \\
\text{DFS}(v)
\end{align*}
\]

Implemented using a global array `Visited` for all recursive calls. `T` is the search tree/forest.

Non-recursive version: based on stacks
Edges classified into two types: $uv \in E$ is a

1. **tree edge**: belongs to $T$
2. **non-tree edge**: does not belong to $T$
Properties of **DFS tree**

**Proposition**

1. $T$ is a forest
2. Connected components of $T$ are same as those of $G$.
3. If $uv \in E$ is a non-tree edge then, in $T$, either:
   1. $u$ is an ancestor of $v$, or
   2. $v$ is an ancestor of $u$.

**Question:** Why are there no cross-edges?
**DFS with Visit Times**

Keep track of when nodes are visited.

\[ \text{DFS}(G) \]

\[
\text{for all } u \in V(G) \text{ do}
\]

\[
\text{Mark } u \text{ as unvisited}
\]

\[
T \text{ is set to } \emptyset
\]

\[
time = 0
\]

\[
\text{while } \exists \text{ unvisited } u \text{ do}
\]

\[
\text{DFS}(u)
\]

Output \( T \)

\[ \text{DFS}(u) \]

\[
\text{Mark } u \text{ as visited}
\]

\[
\text{pre}(u) = ++\text{time}
\]

\[
\text{for each } uv \text{ in } \text{Out}(u) \text{ do}
\]

\[
\text{if } v \text{ is not marked then}
\]

\[
\text{add edge } uv \text{ to } T
\]

\[
\text{DFS}(v)
\]

\[
\text{post}(u) = ++\text{time}
\]
Example

The set of connected components of a graph is the set \( \{c(u) | u \in V\} \).

The connected components in the above graph are \( \{1, 2, 3, 4, 5, 6, 7, 8\} \) and \( \{9, 10\} \).

A graph is said to be connected when it has exactly one connected component. In other words, every pair of vertices in the graph are connected.
pre and post numbers

Node $u$ is active in time interval $[\text{pre}(u), \text{post}(u)]$

Proposition

For any two nodes $u$ and $v$, the two intervals $[\text{pre}(u), \text{post}(u)]$ and $[\text{pre}(v), \text{post}(v)]$ are disjoint or one is contained in the other.
Node $u$ is **active** in time interval $[\text{pre}(u), \text{post}(u)]$

**Proposition**

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**Proof.**
**pre and post numbers**

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**Proof.**

- Assume without loss of generality that $\text{pre}(u) < \text{pre}(v)$. Then $v$ visited after $u$. 
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**Proof.**

- Assume without loss of generality that $\text{pre}(u) < \text{pre}(v)$. Then $v$ visited after $u$.
- If $\text{DFS}(v)$ invoked before $\text{DFS}(u)$ finished, $\text{post}(v) < \text{post}(u)$. 
The node $u$ is active in time interval $[\text{pre}(u), \text{post}(u)]$.

**Proposition**

For any two nodes $u$ and $v$, the two intervals $[\text{pre}(u), \text{post}(u)]$ and $[\text{pre}(v), \text{post}(v)]$ are disjoint or one is contained in the other.

**Proof.**

- Assume without loss of generality that $\text{pre}(u) < \text{pre}(v)$. Then $v$ visited after $u$.
- If $\text{DFS}(v)$ invoked before $\text{DFS}(u)$ finished, $\text{post}(v) < \text{post}(u)$.
- If $\text{DFS}(v)$ invoked after $\text{DFS}(u)$ finished, $\text{pre}(v) > \text{post}(u)$. □
**Pre** and **post** numbers

Node $u$ is **active** in time interval $[\text{pre}(u), \text{post}(u)]$

**Proposition**

*For any two nodes $u$ and $v$, the two intervals $[\text{pre}(u), \text{post}(u)]$ and $[\text{pre}(v), \text{post}(v)]$ are disjoint or one is contained in the other.*

**Proof.**

- Assume without loss of generality that $\text{pre}(u) < \text{pre}(v)$. Then $v$ visited after $u$.
- If $\text{DFS}(v)$ invoked before $\text{DFS}(u)$ finished, $\text{post}(v) < \text{post}(u)$.
- If $\text{DFS}(v)$ invoked after $\text{DFS}(u)$ finished, $\text{pre}(v) > \text{post}(u)$.

**pre** and **post** numbers useful in several applications of **DFS**
DFS in Directed Graphs

DFS(G)
Mark all nodes u as unvisited
T is set to ∅
time = 0
while there is an unvisited node u do
    DFS(u)
Output T

DFS(u)
Mark u as visited
pre(u) = ++time
for each edge (u, v) in Out(u) do
    if v is not visited
        add edge (u, v) to T
        DFS(v)
post(u) = ++time
Example

A directed graph (also called a digraph) is $G = (V, E)$, where $V$ is a set of vertices or nodes, $E \subseteq V \times V$ is the set of ordered pairs of vertices called edges.
DFS Properties

Generalizing ideas from undirected graphs:

1. **DFS**\( (G) \) takes \( O(m + n) \) time.

Edges added form a branching: a forest of out-trees. Output of **DFS**\( (G) \) depends on the order in which vertices are considered.

If \( u \) is the first vertex considered by **DFS**\( (G) \) then **DFS**\( (u) \) outputs a directed out-tree \( T \) rooted at \( u \) and a vertex \( v \) is in \( T \) if and only if \( v \in \text{rch}(u) \).

For any two vertices \( x, y \), the intervals \( [\text{pre}(x), \text{post}(x)] \) and \( [\text{pre}(y), \text{post}(y)] \) are either disjoint or one is contained in the other.

Note: Not obvious whether **DFS**\( (G) \) is useful in directed graphs but it is.
DFS Properties

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1. **DFS**(*G*) takes \(O(m + n)\) time.

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DFS Properties

Generalizing ideas from undirected graphs:

1. **DFS**($G$) takes $O(m + n)$ time.

2. Edges added form a *branching*: a forest of out-trees. Output of **DFS**($G$) depends on the order in which vertices are considered.

3. If $u$ is the first vertex considered by **DFS**($G$) then **DFS**($u$) outputs a directed out-tree $T$ rooted at $u$ and a vertex $v$ is in $T$ if and only if $v \in rch(u)$.
DFS Properties

Generalizing ideas from undirected graphs:

1. **DFS(G)** takes $O(m + n)$ time.

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4. For any two vertices $x, y$ the intervals $[\text{pre}(x), \text{post}(x)]$ and $[\text{pre}(y), \text{post}(y)]$ are either disjoint or one is contained in the other.
DFS Properties

Generalizing ideas from undirected graphs:

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**Note:** Not obvious whether **DFS(G)** is useful in dir graphs but it is.
DFS Tree

Edges of $G$ can be classified with respect to the DFS tree $T$ as:

1. **Tree edges** that belong to $T$
   
2. A **forward edge** is a non-tree edges $(x, y)$ such that $\text{pre}(x) < \text{pre}(y) < \text{post}(y) < \text{post}(x)$.

3. A **backward edge** is a non-tree edge $(y, x)$ such that $\text{pre}(x) < \text{pre}(y) < \text{post}(y) < \text{post}(x)$.

4. A **cross edge** is a non-tree edges $(x, y)$ such that the intervals $[\text{pre}(x), \text{post}(x)]$ and $[\text{pre}(y), \text{post}(y)]$ are disjoint.
Types of Edges

- A
- B
- C
- D

- Forward
- Backward
- Cross
Cycles in graphs

**Question:** Given an *undirected* graph how do we check whether it has a cycle and output one if it has one?

**Question:** Given an *directed* graph how do we check whether it has a cycle and output one if it has one?
Using DFS…
... to check for Acyclicity and compute Topological Ordering

**Question**

Given $G$, is it a **DAG**? If it is, generate a topological sort. Else output a cycle $C$. 

**DFS based algorithm:**

1. Compute $\text{DFS}(G)$
2. If there is a back edge $e = (v, u)$ then $G$ is not a DAG. Output cycle $C$ formed by path from $u$ to $v$ in $T$ plus edge $(v, u)$.
3. Otherwise output nodes in decreasing post-visit order. Note: no need to sort, $\text{DFS}(G)$ can output nodes in this order.

Algorithm runs in $O(n + m)$ time.

Correctness is not so obvious. See next two propositions.
Using DFS...

... to check for Acylicity and compute Topological Ordering

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Algorithm runs in $O(n + m)$ time.
Correctness is not so obvious. See next two propositions.
Back edge and Cycles

Proposition

\( G \) has a cycle iff there is a back-edge in \( \text{DFS}(G) \).

Proof.

If: \((u, v)\) is a back edge implies there is a cycle \(C\) consisting of the path from \(v\) to \(u\) in \(\text{DFS}\) search tree and the edge \((u, v)\).

Only if: Suppose there is a cycle \(C = v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k \rightarrow v_1\). Let \(v_i\) be first node in \(C\) visited in \(\text{DFS}\). All other nodes in \(C\) are descendants of \(v_i\) since they are reachable from \(v_i\).

Therefore, \((v_{i-1}, v_i)\) (or \((v_k, v_1)\) if \(i = 1\)) is a back edge.
Proof

Proposition

If $G$ is a DAG and $\text{post}(v) > \text{post}(u)$, then $(u, v)$ is not in $G$.

Proof.

Assume $\text{post}(v) > \text{post}(u)$ and $(u, v)$ is an edge in $G$. We derive a contradiction. One of two cases holds from DFS property.

- **Case 1**: $[\text{pre}(u), \text{post}(u)]$ is contained in $[\text{pre}(v), \text{post}(v)]$. Implies that $u$ is explored during DFS($v$) and hence is a descendent of $v$. Edge $(u, v)$ implies a cycle in $G$ but $G$ is assumed to be DAG!

- **Case 2**: $[\text{pre}(u), \text{post}(u)]$ is disjoint from $[\text{pre}(v), \text{post}(v)]$. This cannot happen since $v$ would be explored from $u$. 


Example

```graphviz
digraph G {
    a -> b -> c;
    d -> e;
    f -> e;
    g -> e;
    h -> e;
    h -> f;
}
```
Part III

Linear time algorithm for finding all strong connected components of a directed graph
Finding all SCCs of a Directed Graph

Problem

Given a directed graph \( G = (V, E) \), output all its strong connected components.
Finding all SCCs of a Directed Graph

Problem
Given a directed graph \( G = (V, E) \), output all its strong connected components.

Straightforward algorithm:

Mark all vertices in \( V \) as not visited.

\[
\text{for each vertex } u \in V \text{ not visited yet do }
\]

find \( SCC(G, u) \) the strong component of \( u \):

Compute \( rch(G, u) \) using \( DFS(G, u) \)
Compute \( rch(G^{\text{rev}}, u) \) using \( DFS(G^{\text{rev}}, u) \)

\( SCC(G, u) \leftarrow rch(G, u) \cap rch(G^{\text{rev}}, u) \)

\( \forall u \in SCC(G, u) : \) Mark \( u \) as visited.

Running time: \( O(n(n + m)) \)
Finding all \textbf{SCCs} of a Directed Graph

\textbf{Problem}

Given a directed graph $G = (V, E)$, output all its strong connected components.

Straightforward algorithm:

Mark all vertices in $V$ as not visited.

\hspace{1cm} \textbf{for} each vertex $u \in V$ not visited yet \textbf{do} \\
\hspace{2cm} find $\text{SCC}(G, u)$ the strong component of $u$:
\hspace{2cm} Compute $\text{rch}(G, u)$ using $\text{DFS}(G, u)$
\hspace{2cm} Compute $\text{rch}(G^{\text{rev}}, u)$ using $\text{DFS}(G^{\text{rev}}, u)$
\hspace{2cm} $\text{SCC}(G, u) \iff \text{rch}(G, u) \cap \text{rch}(G^{\text{rev}}, u)$
\hspace{2cm} $\forall u \in \text{SCC}(G, u)$: Mark $u$ as visited.

Running time: $O(n(n + m))$

Is there an $O(n + m)$ time algorithm?
### Structure of a Directed Graph

**Reminder**

$G^{\text{SCC}}$ is created by collapsing every strong connected component to a single vertex.

**Proposition**

*For a directed graph $G$, its meta-graph $G^{\text{SCC}}$ is a DAG.*
Linear-time Algorithm for SCCs: Ideas

Exploit structure of meta-graph...

### Wishful Thinking Algorithm

1. Let $u$ be a vertex in a sink SCC of $G^{SCC}$
2. Do $\text{DFS}(u)$ to compute $\text{SCC}(u)$
3. Remove $\text{SCC}(u)$ and repeat
Linear-time Algorithm for SCCs: Ideas

Exploit structure of meta-graph...

### Wishful Thinking Algorithm

1. Let $u$ be a vertex in a sink SCC of $G^{SCC}$
2. Do $DFS(u)$ to compute $SCC(u)$
3. Remove $SCC(u)$ and repeat

### Justification

1. $DFS(u)$ only visits vertices (and edges) in $SCC(u)$
Linear-time Algorithm for SCCs: Ideas

Exploit structure of meta-graph...

**Wishful Thinking Algorithm**

1. Let $u$ be a vertex in a sink SCC of $G^{SCC}$
2. Do $DFS(u)$ to compute $SCC(u)$
3. Remove $SCC(u)$ and repeat

**Justification**

1. $DFS(u)$ only visits vertices (and edges) in $SCC(u)$
2. ... since there are no edges coming out of a sink!
### Linear-time Algorithm for SCCs: Ideas

Exploit structure of meta-graph...

#### Wishful Thinking Algorithm

1. Let $u$ be a vertex in a *sink* SCC of $G^{SCC}$
2. Do $\text{DFS}(u)$ to compute $\text{SCC}(u)$
3. Remove $\text{SCC}(u)$ and repeat

#### Justification

1. $\text{DFS}(u)$ only visits vertices (and edges) in $\text{SCC}(u)$
2. ... since there are no edges coming out a sink!
3. $\text{DFS}(u)$ takes time proportional to size of $\text{SCC}(u)$
Linear-time Algorithm for SCCs: Ideas

Exploit structure of meta-graph...

### Wishful Thinking Algorithm

1. Let $u$ be a vertex in a sink SCC of $G^{\text{SCC}}$
2. Do $\text{DFS}(u)$ to compute $\text{SCC}(u)$
3. Remove $\text{SCC}(u)$ and repeat

### Justification

1. $\text{DFS}(u)$ only visits vertices (and edges) in $\text{SCC}(u)$
2. ... since there are no edges coming out a sink!
3. $\text{DFS}(u)$ takes time proportional to size of $\text{SCC}(u)$
4. Therefore, total time $O(n + m)$!
Big Challenge(s)

How do we find a vertex in a sink $SCC$ of $G^{SCC}$?

Answer: DFS$(G)$ gives some information!
Big Challenge(s)

How do we find a vertex in a sink $\text{SCC}$ of $G^{\text{SCC}}$?

Can we obtain an *implicit* topological sort of $G^{\text{SCC}}$ without computing $G^{\text{SCC}}$?
Big Challenge(s)

How do we find a vertex in a sink SCC of $G^{SCC}$?

Can we obtain an implicit topological sort of $G^{SCC}$ without computing $G^{SCC}$?

Answer: DFS($G$) gives some information!
Linear Time Algorithm

...for computing the strong connected components in $G$

```
do DFS(G_{rev}) and output vertices in decreasing post order.
Mark all nodes as unvisited
for each $u$ in the computed order do
  if $u$ is not visited then
    DFS(u)
  Let $S_u$ be the nodes reached by $u$
  Output $S_u$ as a strong connected component
  Remove $S_u$ from $G$
```

**Theorem**

Algorithm runs in time $O(m + n)$ and correctly outputs all the SCCs of $G$. 
Graph $G$:

$G\xrightarrow{\Rightarrow}G^{\text{rev}}$:

DFS of reverse graph:

Pre/Post DFS numbering of reverse graph:
Removing connected components: 1

Original graph $G$ with rev post numbers:

Do **DFS** from vertex $G$ remove it.

SCC computed: \{ $G$ \}
Linear Time Algorithm: An Example

Removing connected components: 2

Do DFS from vertex $G$
remove it.

$\text{SCC computed: } \{G\}$

Do DFS from vertex $H$, remove it.

$\text{SCC computed: } \{G\}, \{H\}$
Linear Time Algorithm: An Example

Removing connected components: 3

Do DFS from vertex $H$, remove it.

$\text{SCC computed: } \{G\}, \{H\}$

Do DFS from vertex $B$
Remove visited vertices: $\{F, B, E\}$.

$\text{SCC computed: } \{G\}, \{H\}, \{F, B, E\}$
Linear Time Algorithm: An Example

Removing connected components: 4

Do **DFS** from vertex *F*
Remove visited vertices: 
\{*F*, *B*, *E*\}.

**SCC** computed:
\{*G*\}, \{*H*\}, \{*F*, *B*, *E*\}

⇒

Do **DFS** from vertex *A*
Remove visited vertices: 
\{*A*, *C*, *D*\}.

**SCC** computed:
\{*G*\}, \{*H*\}, \{*F*, *B*, *E*\}, \{*A*, *C*, *D*\}
Linear Time Algorithm: An Example

Final result

SCC computed:
{G}, {H}, {F, B, E}, {A, C, D}
Which is the correct answer!
Obtaining the meta-graph...

Once the strong connected components are computed.

Exercise:

Given all the strong connected components of a directed graph \( G = (V, E) \) show that the meta-graph \( G^{SCC} \) can be obtained in \( O(m + n) \) time.
Solving Problems on Directed Graphs

A template for a class of problems on directed graphs:

- Is the problem solvable when $G$ is strongly connected?
- Is the problem solvable when $G$ is a DAG?
- If the above two are feasible then is the problem solvable in a general directed graph $G$ by considering the meta graph $G^{SCC}$?
Part IV

An Application to make
I know what make/makefile is.

I do NOT know what make/makefile is.
make Utility [Feldman]

1. Unix utility for automatically building large software applications
2. A makefile specifies
   1. Object files to be created,
   2. Source/object files to be used in creation, and
   3. How to create them
An Example makefile

project:  main.o utils.o command.o
         cc -o project main.o utils.o command.o

main.o:  main.c defs.h
         cc -c main.c

utils.o: utils.c defs.h command.h
         cc -c utils.c

command.o: command.c defs.h command.h
           cc -c command.c
makefile as a Digraph

main.c → main.o
util.c → main.o
defs.h → util.o → project
command.h → command.o
command.c
Computational Problems for make

1. Is the makefile reasonable?
2. If it is reasonable, in what order should the object files be created?
3. If it is not reasonable, provide helpful debugging information.
4. If some file is modified, find the fewest compilations needed to make application consistent.
Algorithms for make

1. Is the makefile reasonable? Is $G$ a DAG?
2. If it is reasonable, in what order should the object files be created? Find a topological sort of a DAG.
3. If it is not reasonable, provide helpful debugging information. Output a cycle. More generally, output all strong connected components.
4. If some file is modified, find the fewest compilations needed to make application consistent.
   - Find all vertices reachable (using DFS/BFS) from modified files in directed graph, and recompile them in proper order. Verify that one can find the files to recompile and the ordering in linear time.
1. Given a directed graph $G$, its SCCs and the associated acyclic meta-graph $G^{SCC}$ give a structural decomposition of $G$ that should be kept in mind.

2. There is a DFS based linear time algorithm to compute all the SCCs and the meta-graph. Properties of DFS crucial for the algorithm.

3. DAGs arise in many application and topological sort is a key property in algorithm design. Linear time algorithms to compute a topological sort (there can be many possible orderings so not unique).