Directed Graphs and DFS

Lecture 16
March 23, 2021

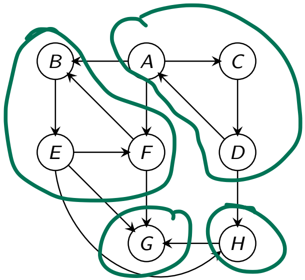
Topics

- Structure of directed graphs
    - Directed acyclic graphs (DAGs) and topological sort
    - Strong connected components and meta graph representation
- **DFS** and its properties includig pre/post numbering
- Linear time algorithm for compuing all SCCs of a directed graph

Strong Connected Components (SCCs)

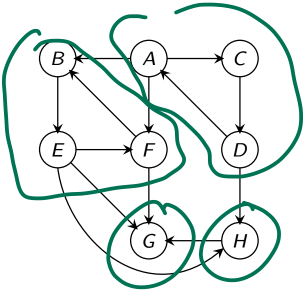
Algorithmic Problem

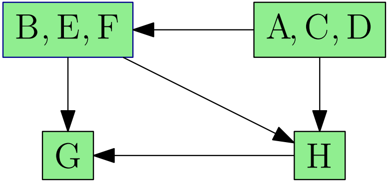
Find all SCCs of a given directed graph.



Previous lecture:
Saw an $O(n \cdot (n + m))$ time algorithm.
This lecture: sketch of a $O(n + m)$ time algorithm.

Graph of SCCs





Graph of SCCs G^{SCC}

Meta-graph of SCCs

Let $S_1, S_2, \ldots S_k$ be the strong connected components (i.e., SCCs) of G. The graph of SCCs is G^{SCC}

1. Vertices are $S_1, S_2, \ldots S_k$
2. There is an edge (S_i, S_j) if there is some $u \in S_i$ and $v \in S_j$ such that (u, v) is an edge in G.

Reversal and SCCs

Proposition

For any graph G, the graph of SCCs of G^{rev} is the same as the reversal of G^{SCC}.

Proof.

Exercise. ☐

SCCs and DAGs

Proposition

For any graph G, the graph G^{SCC} has no directed cycle.

Proof.

If G^{SCC} has a cycle $S_1, S_2, \ldots, S_k$ then $S_1 \cup S_2 \cup \cdots \cup S_k$ should be in the same SCC in G. Formal details: exercise. ☐

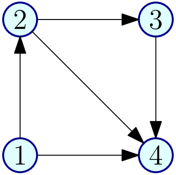
Part I

Directed Acyclic Graphs

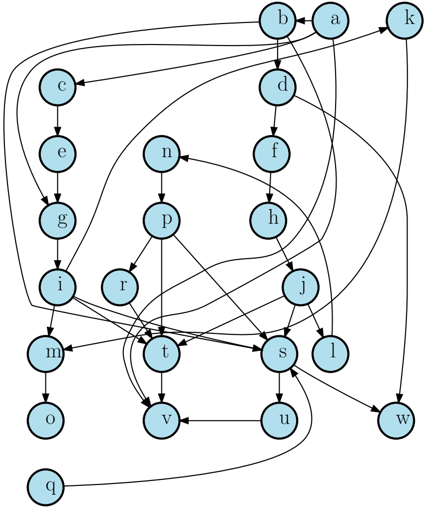
Directed Acyclic Graphs

Definition

A directed graph G is a **directed acyclic graph** (DAG) if there is no directed cycle in G.



DAGs can be complicated



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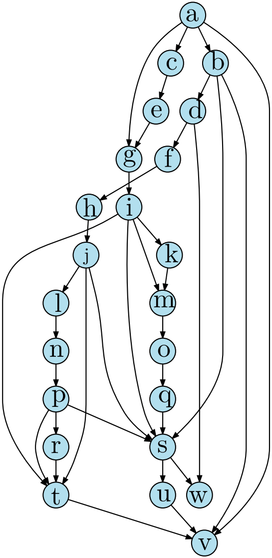
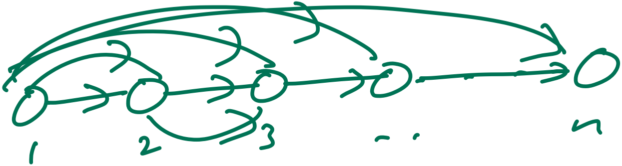
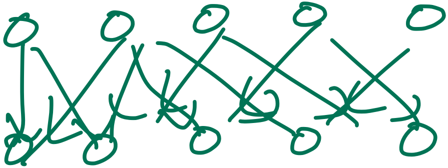


DAGs can be dense

A DAG on n vertices can have $\Omega(n^2)$ edges. Which ones?



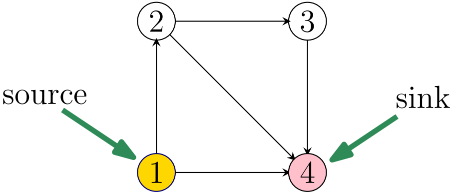
$\Omega(n^2)$.



$\frac{n}{2}$

$\frac{n}{2}$

Sources and Sinks



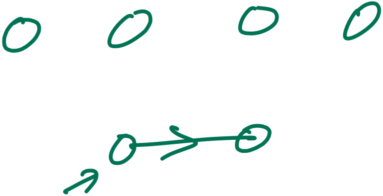
Definition

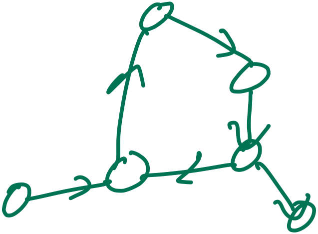
1. A vertex ***u*** is a **source** if it has no in-coming edges.
2. A vertex ***u*** is a **sink** if it has no out-going edges.

DAG Properties

Proposition

Every DAG G has at least one source and at least one sink.





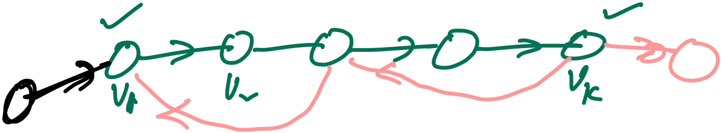
DAG Properties

Proposition

Every DAG G has at least one source and at least one sink.

Proof.

Let $P = v_1, v_2, \ldots, v_k$ be a *longest* (or a *maximal*) path in G. Claim that v_1 is a source and v_k is a sink. Suppose v_1 is not a source. Then v_1 has an incoming edge which either creates a cycle, or creates a longer path, both of which are contradictions. Similarly if v_k is not a sink, then it has an outgoing edge and creates a cycle, or a longer path. □



DAG Properties

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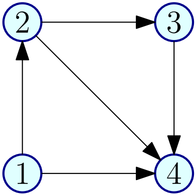
1. G is a DAG if and only if G^{rev} is a DAG.
2. G is a DAG if and only each node is in its own strong connected component.

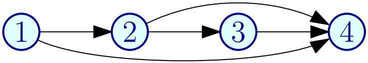
Formal proofs: exercise.

Topological Ordering/Sorting



Graph G

0 0 0 0 0 0
1 2 1 2 2 1



Topological Ordering of G

Definition

A **topological ordering**/**topological sorting** of $G = (V, E)$ is an ordering $\prec$ on V such that if $(u \rightarrow v) \in E$ then $u \prec v$.

Informal equivalent definition:

One can order the vertices of the graph along a line (say the x-axis) such that all edges are from left to right.

DAGs and Topological Sort

Lemma

A directed graph G can be topologically ordered **if and only if** *it is a* DAG.

Need to show both directions.

DAGs and Topological Sort

Lemma

A directed graph G can be topologically ordered **if** *it is a* DAG.

Proof.

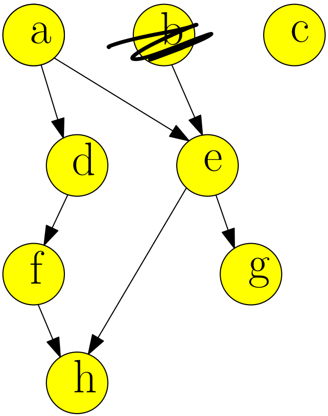
Consider the following algorithm:

1. Pick a source u, output it.
2. Remove u and all edges out of u.
3. Repeat until graph is empty.

Exercise: prove this gives toplogical sort. ☐

Exercise: show algorithm can be implemented in $O(m + n)$ time.

Topological Sort: Example



b e c
a d g

a b c d e f g h

DAGs and Topological Sort

Lemma

A directed graph G can be topologically ordered **only if** *it is a DAG.*

Proof.

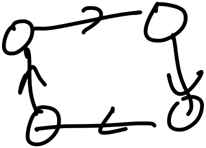
Suppose G is not a DAG and has a topological ordering $\prec$. G has a cycle $C = u_1, u_2, \ldots, u_k, u_1$.

Then $u_1 \prec u_2 \prec \ldots \prec u_k \prec u_1$!

That is... $u_1 \prec u_1$.

A contradiction (to $\prec$ being an order).

Not possible to topologically order the vertices. $\square$



DAGs and Topological Sort

Note: A DAG G may have many different topological sorts.

Question: What is a DAG with the most number of distinct topological sorts for a given number *n* of vertices?

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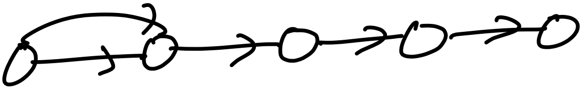
$n!$



①

Cycles in graphs

Question: Given an *undirected* graph how do we check whether it has a cycle and output one if it has one?

Question: Given an *directed* graph how do we check whether it has a cycle and output one if it has one?

To Remember: Structure of Graphs

Undirected graph: connected components of $G = (V, E)$ partition V and can be computed in $O(m + n)$ time.

Directed graph: the meta-graph G^{SCC} of G can be computed in $O(m + n)$ time. G^{SCC} gives information on the partition of V into strong connected components and how they form a DAG structure.

Above structural decomposition will be useful in several algorithms

Part II

Depth First Search (DFS)

Depth First Search

DFS is a special case of Basic Search but is a versatile graph exploration strategy. John Hopcroft and Bob Tarjan (Turing Award winners) demonstrated the power of **DFS** to understand graph structure. **DFS** can be used to obtain linear time ($O(m+n)$) algorithms for

1. Finding cut-edges and cut-vertices of undirected graphs
2. Finding strong connected components of directed graphs
3. Linear time algorithm for testing whether a graph is planar

Many other applications as well.

DFS in Undirected Graphs

Recursive version. Easier to understand some properties.

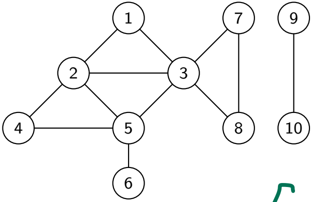
```
DFS($G$)
    for all $u \in V(G)$ do
        Mark $u$ as unvisited
        Set pred($u$) to null
    $T$ is set to $\emptyset$
    while $\exists$ unvisited $u$ do
        DFS($u$)
    Output $T$
```

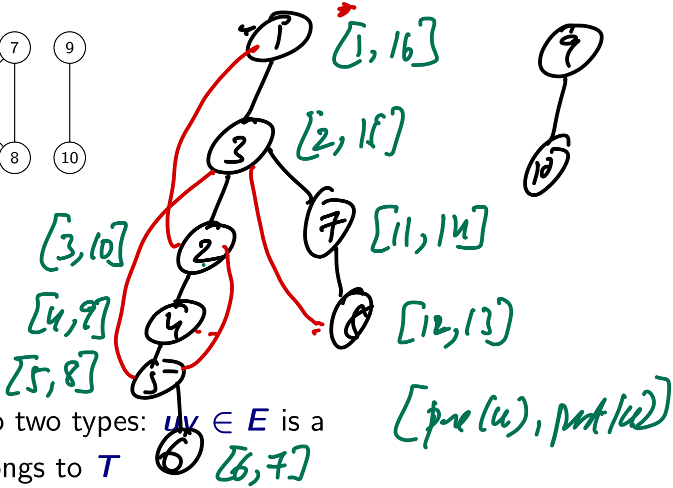
```
DFS($u$)
    Mark $u$ as visited
    for each $uv$ in $Out(u)$ do
        if $v$ is not visited then
            add edge $uv$ to $T$
            set pred($v$) to $u$
            DFS($v$)
```

Implemented using a global array ***Visited*** for all recursive calls.
T is the search tree/forest.

Non-recursive version: based on stacks

Example





Edges classified into two types: $uv \in E$ is a

1. tree edge: belongs to T
2. non-tree edge: does not belong to T

Properties of DFS tree

Proposition

1. *T is a forest*
2. *connected components of T are same as those of G.*
3. *If $uv \in E$ is a non-tree edge then, in T, either:*
    1. *u is an ancestor of v, or*
    2. *v is an ancestor of u.*

Question: Why are there no *cross-edges*?

DFS with Visit Times

Keep track of when nodes are visited.

```
DFS($G$)
    for all $u \in V(G)$ do
        Mark $u$ as unvisited
    $T$ is set to $\emptyset$
    $time = 0$
    while $\exists$unvisited $u$ do
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```
DFS($u$)
    Mark $u$ as visited
    $\mathrm{pre}(u) = {++}time$
    for each $uv$ in $Out(u)$ do
        if $v$ is not marked then
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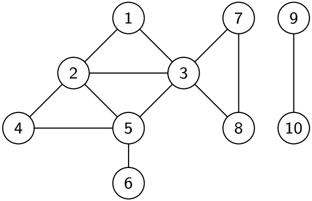
```

Example



pre and post numbers

Node u is **active** in time interval $[\mathrm{pre}(u), \mathrm{post}(u)]$

Proposition

For any two nodes u *and* v*, the two intervals* $[\mathrm{pre}(u), \mathrm{post}(u)]$ *and* $[\mathrm{pre}(v), \mathrm{post}(v)]$ *are disjoint or one is contained in the other.*

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Proof.

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pre and post numbers useful in several applications of **DFS**

DFS in Directed Graphs

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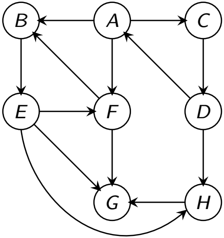
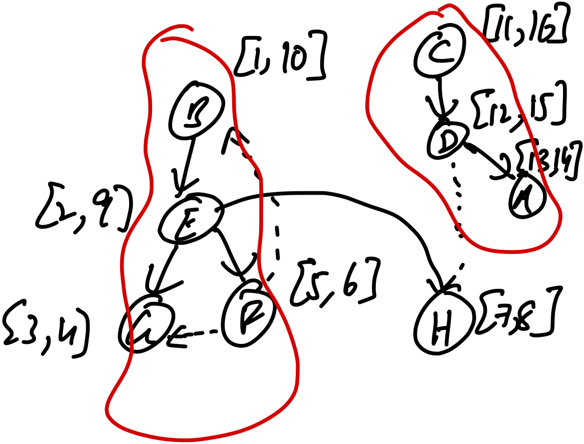
Example





DFS Properties

Generalizing ideas from undirected graphs:

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4. For any two vertices x, y the intervals $[\mathbf{pre}(x), \mathbf{post}(x)]$ and $[\mathbf{pre}(y), \mathbf{post}(y)]$ are either disjoint or one is contained in the other.

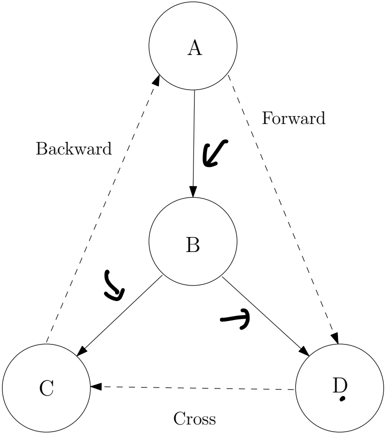
Note: Not obvious whether **DFS**(G) is useful in dir graphs but it is.

DFS Tree

Edges of G can be classified with respect to the **DFS** tree T as:

1. **Tree edges** that belong to T
2. A **forward edge** is a non-tree edges (x, y) such that $\mathbf{pre}(x) < \mathbf{pre}(y) < \mathbf{post}(y) < \mathbf{post}(x)$.
3. A **backward edge** is a non-tree edge (y, x) such that $\mathbf{pre}(x) < \mathbf{pre}(y) < \mathbf{post}(y) < \mathbf{post}(x)$.
4. A **cross edge** is a non-tree edges (x, y) such that the intervals $[\mathbf{pre}(x), \mathbf{post}(x)]$ and $[\mathbf{pre}(y), \mathbf{post}(y)]$ are disjoint.

Types of Edges



tree edge

Cycles in graphs

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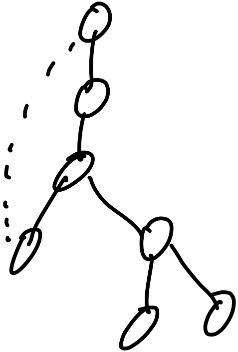
Question: Given an *directed* graph how do we check whether it has a cycle and output one if it has one?



Using DFS...

... to check for Acylicity and compute Topological Ordering

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Given G, is it a DAG? If it is, generate a topological sort. Else output a cycle C.

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DFS based algorithm:

1. Compute **DFS**(G)
2. If there is a back edge $e = (v, u)$ then G is not a DAG. Output cyclce C formed by path from u to v in T plus edge (v, u).
3. Otherwise output nodes in decreasing post-visit order. Note: no need to sort, $DFS(G)$ can output nodes in this order.

Algorithm runs in $O(n + m)$ time.

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Algorithm runs in $O(n + m)$ time.
Correctness is not so obvious. See next two propositions.

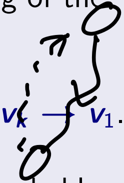
Back edge and Cycles

Proposition

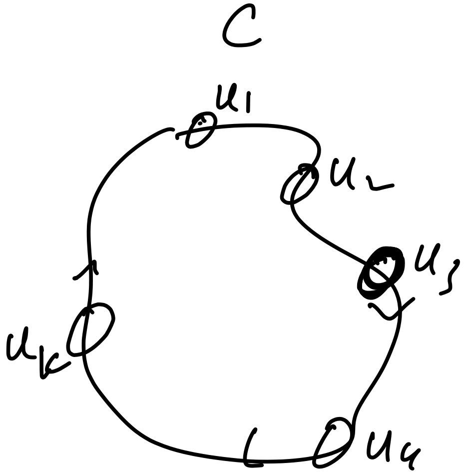
*G has a cycle iff there is a back-edge in **DFS**(G).*

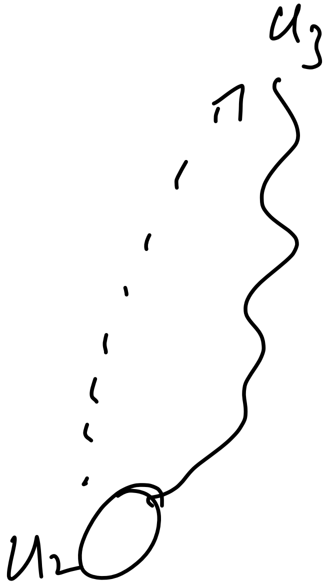
Proof.

If: (u, v) is a back edge implies there is a cycle C consisting of the path from v to u in **DFS** search tree and the edge (u, v).



Only if: Suppose there is a cycle $C = v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k \rightarrow v_1$.
Let v_i be first node in C visited in **DFS**.
All other nodes in C are descendants of v_i since they are reachable from v_i.
Therefore, (v_{i-1}, v_i) (or (v_k, v_1) if $i = 1$) is a back edge. ☐


C
u_1
u_2
u_3
u_4
u_k


u_3
u_2

Proof

Proposition

If G is a DAG *and* $\mathbf{post}(v) > \mathbf{post}(u)$*, then* (u, v) *is not in* G*.*

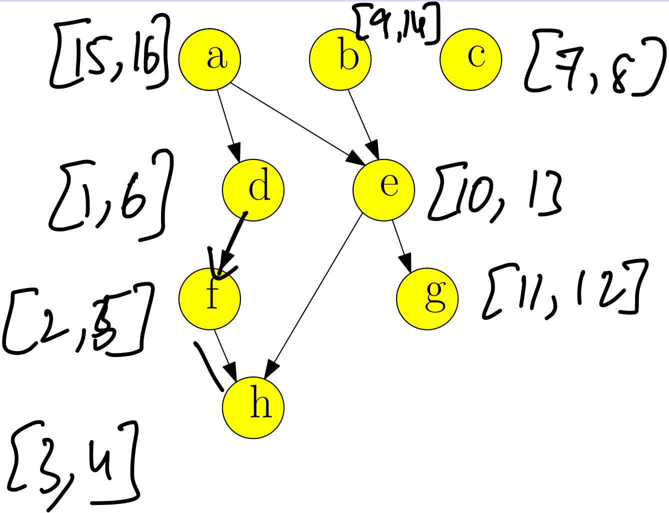
Proof.

Assume $\mathbf{post}(v) > \mathbf{post}(u)$ *and* (u, v) is an edge in G. We derive a contradiction. One of two cases holds from DFS property.

- Case 1: $[pre(u), post(u)]$ is contained in $[pre(v), post(v)]$. Implies that u is explored during **DFS**(v) and hence is a descendent of v. Edge (u, v) implies a cycle in G but G is assumed to be DAG!
- Case 2: $[pre(u), post(u)]$ is disjoint from $[pre(v), post(v)]$. This cannot happen since v would be explored from u.

□

Example



a b e g c d f h

Part III

Linear time algorithm for finding all strong connected components of a directed graph

Finding all SCCs of a Directed Graph

Problem

Given a directed graph $G = (V, E)$, output *all* its strong connected components.

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Straightforward algorithm:

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Mark all vertices in $V$ as not visited.
for each vertex $u \in V$ not visited yet do
    find SCC($G, u$) the strong component of $u$:
        Compute $\mathsf{rch}(G, u)$ using $DFS(G, u)$
        Compute $\mathsf{rch}(G^{\mathrm{rev}}, u)$ using $DFS(G^{\mathrm{rev}}, u)$
        $\mathrm{SCC}(G, u) \Leftarrow \mathsf{rch}(G, u) \cap \mathsf{rch}(G^{\mathrm{rev}}, u)$
        $\forall u \in \mathrm{SCC}(G, u)$: Mark $u$ as visited.
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Running time: $O(n(n + m))$

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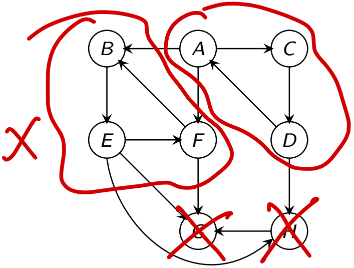
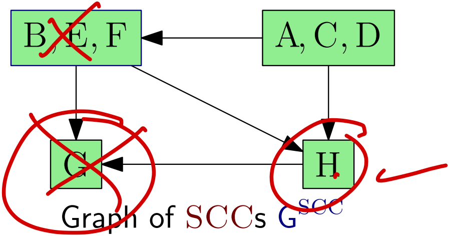
Running time: $O(n(n + m))$

Is there an $O(n + m)$ time algorithm?

Structure of a Directed Graph





Graph of SCCs G^{SCC}

Reminder

G^{SCC} is created by collapsing every strong connected component to a single vertex.

Proposition

For a directed graph G, its meta-graph G^{SCC} is a DAG.

Linear-time Algorithm for SCCs: Ideas

Exploit structure of meta-graph...

Wishful Thinking Algorithm

1. Let u be a vertex in a *sink* SCC of G^{SCC}
2. Do **DFS**(u) to compute $\mathrm{SCC}(u)$
3. Remove $\mathrm{SCC}(u)$ and repeat

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4.

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2. ... since there are no edges coming out a sink!
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4. Therefore, total time $O(n + m)$!

Big Challenge(s)

How do we find a vertex in a sink SCC of G^{SCC}?

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Can we obtain an *implicit* topological sort of G^{SCC} without computing G^{SCC}?

Answer: **DFS**(G) gives some information!

Linear Time Algorithm

...for computing the strong connected components in G

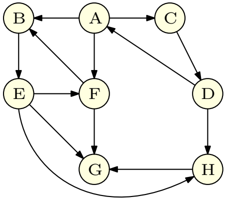
```
do DFS($G^{\text{rev}}$) and output vertices in decreasing post order.
Mark all nodes as unvisited
for each $u$ in the computed order do
    if $u$ is not visited then
        DFS($u$)
        Let $S_u$ be the nodes reached by $u$
        Output $S_u$ as a strong connected component
        Remove $S_u$ from G
```

Theorem

Algorithm runs in time $O(m + n)$ and correctly outputs all the SCCs of G.

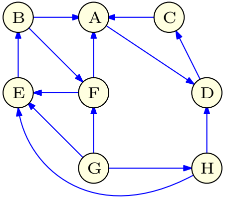
Linear Time Algorithm: Initial steps

Graph G:



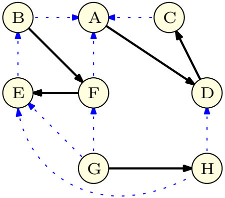
⟹

Reverse graph G^{rev}:



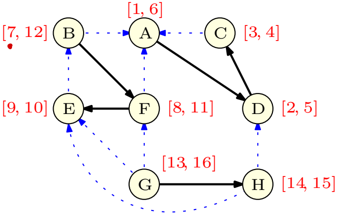
⟹

DFS of reverse graph:



⟹

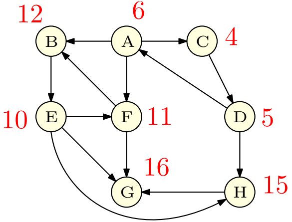
Pre/Post **DFS** numbering of reverse graph:



Linear Time Algorithm: An Example

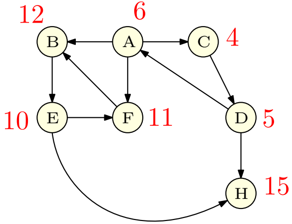
Removing connected components: 1

Original graph G with rev post numbers:



⟹

Do **DFS** from vertex G remove it.



SCC computed:

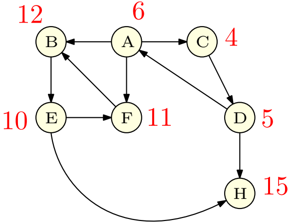
$\{G\}$

G, H, B, E, F, A, D, C

Linear Time Algorithm: An Example

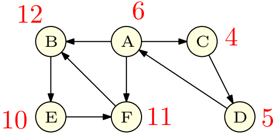
Removing connected components: 2

Do **DFS** from vertex G remove it.



$\Longrightarrow$

Do **DFS** from vertex H, remove it.



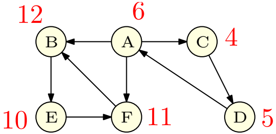
SCC computed:
$\{G\}$

SCC computed:
$\{G\}, \{H\}$

Linear Time Algorithm: An Example

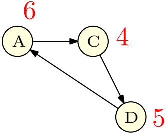
Removing connected components: 3

Do **DFS** from vertex H, remove it.



$\Longrightarrow$

Do **DFS** from vertex B
Remove visited vertices:
$\{F, B, E\}$.



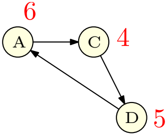
SCC computed:
$\{G\}, \{H\}$

SCC computed:
$\{G\}, \{H\}, \{F, B, E\}$

Linear Time Algorithm: An Example

Removing connected components: 4

Do **DFS** from vertex F
Remove visited vertices:
$\{F, B, E\}$.



$\Longrightarrow$

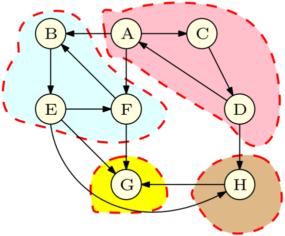
Do **DFS** from vertex A
Remove visited vertices:
$\{A, C, D\}$.

SCC computed:
$\{G\}, \{H\}, \{F, B, E\}$

SCC computed:
$\underline{\{G\}}, \underline{\{H\}}, \underline{\{F, B, E\}}, \underline{\{A, C, D\}}$

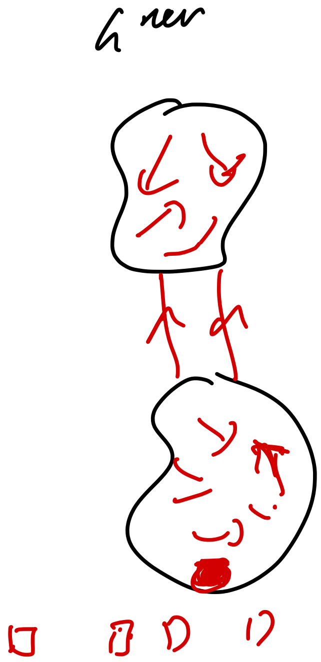
Linear Time Algorithm: An Example

Final result



SCC computed:
$\{G\}, \{H\}, \{F, B, E\}, \{A, C, D\}$
Which is the correct answer!


G.
S
S2


G new

Obtaining the meta-graph...

Once the strong connected components are computed.

Exercise:

Given all the strong connected components of a directed graph $G = (V, E)$ show that the meta-graph $\mathsf{G}^{\mathrm{SCC}}$ can be obtained in $O(m + n)$ time.

Solving Problems on Directed Graphs

A template for a class of problems on directed graphs:

- Is the problem solvable when G is strongly connected?
- Is the problem solvable when G is a DAG?
- If the above two are feasible then is the problem solvable in a general directed graph G by considering the meta graph G^{SCC}?

Part IV

An Application to `make`

Make/Makefile

- (A) I know what make/makefile is.
- (B) I do NOT know what make/makefile is.

`make` Utility [Feldman]

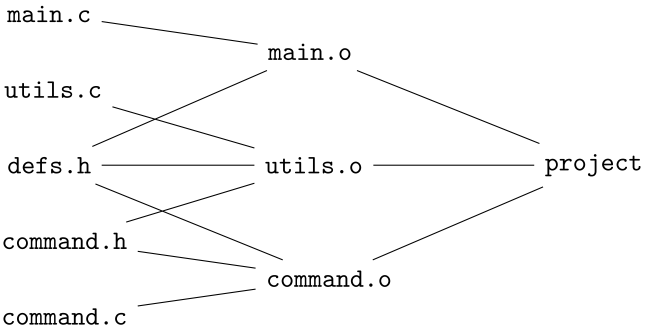
1. Unix utility for automatically building large software applications
2. A makefile specifies
    1. Object files to be created,
    2. Source/object files to be used in creation, and
    3. How to create them

An Example `makefile`

```
project:  main.o utils.o command.o
    cc -o project main.o utils.o command.o

main.o:  main.c defs.h
    cc -c main.c
utils.o:  utils.c defs.h command.h
    cc -c utils.c
command.o:  command.c defs.h command.h
    cc -c command.c
```

`makefile` as a Digraph



Computational Problems for `make`

1. Is the `makefile` reasonable?
2. If it is reasonable, in what order should the object files be created?
3. If it is not reasonable, provide helpful debugging information.
4. If some file is modified, find the fewest compilations needed to make application consistent.

Algorithms for `make`

1. Is the `makefile` reasonable? Is G a DAG?
2. If it is reasonable, in what order should the object files be created? Find a topological sort of a DAG.
3. If it is not reasonable, provide helpful debugging information. Output a cycle. More generally, output all strong connected components.
4. If some file is modified, find the fewest compilations needed to make application consistent.
    1. Find all vertices reachable (using **DFS**/**BFS**) from modified files in directed graph, and recompile them in proper order. Verify that one can find the files to recompile and the ordering in linear time.

Take away Points

1. Given a directed graph G, its SCCs and the associated acyclic meta-graph G^{SCC} give a structural decomposition of G that should be kept in mind.
2. There is a **DFS** based linear time algorithm to compute all the SCCs and the meta-graph. Properties of **DFS** crucial for the algorithm.
3. DAGs arise in many application and topological sort is a key property in algorithm design. Linear time algorithms to compute a topological sort (there can be many possible orderings so not unique).