Given a language $L$ over an arbitrary alphabet $\Sigma$, we can define the language

$$\boxed{\textsc{Cycle}(L) := \{yx \mid xy \in L\}}$$

We can then ask the question, if $L$ is regular, is $\textsc{Cycle}(L)$ also regular? The answer is ***yes***. Since $L$ is regular, there exists some DFA $M = (Q, \delta, s, A)$ such that $L = L(M)$. We will construct a new **NFA** $N$ from $M$ such that $N$ accepts $\textsc{Cycle}(L)$.

Given a string $w$, we need $N$ to accept $w$ if and only if $w \in \textsc{Cycle}(L)$. This means that there exists some way to split $w$ into two substrings $y$ and $x$ such that $w = yx$ and $xy \in L$. We are immediately confronted with two problems:

(i) There are $|w| + 1$ ways to split $w$ into $y$ and $x$, and we do not know which of these ways result in $xy$ being a string in $L$.

(ii) Even if there is a way to split $w$ into $w = yx$ such that $xy \in L$, we still need to contend with the fact that we are not reading $xy$, but rather $yx$, and furthermore since we are reading the input string one character at a time, we know nothing about how many characters belong to the $y$ part (or how many characters belong to the $x$ part) ahead of time.

We will get around both issues using non-determinism.

Here's the idea. Suppose that $w \in \textsc{Cycle}(L)$. We would *like* to guess a decomposition $w = yx$ so that $xy \in L$, and then feed $xy$ through $M$ to verify that $xy \in L$. But we can't, since we don't know anything about $w$ yet. But we can try to simulate and guess some other things.

- Suppose we could *guess* the state $q_x := \delta^*(s, x)$ (i.e, the state we land on in $M$ if, starting from $s$, we read in $x$). If we are correct, then starting at $q_x$ and reading the first $|y|$ characters of $w$ (i.e., $y$) gets us to an accepting state, since $\delta^*(q_x, y) = \delta^*(\delta^*(s, x), y) = \delta^*(s, xy)$ which as an accepting state by the assumption $xy \in L$.

- Since we don't actually know $y$, we can do the following instead. Suppose that, starting from $q_x$, we read some number of characters from $w$ and end at an accepting state. At this point, we could *guess* that we actually correctly read $y$.

- At this point, we have made two guesses: the state $q_x$, and if we are done reading $y$. But how do we know that our two guesses were actually correct?

  Well, suppose we go back to the starting state $s$ and read in the rest of $w$. If our two guesses were correct, then the rest of the input is $x$, and reading the rest of the input should get us to $\delta^*(s, x) = q_x$.

Well, how do we actually do any of that? Well, we will build an NFA that *simulates* doing the above. This NFA needs to keep track of at least the following pieces of information: the state we guessed to be $q_x$, whether or not we have guessed that we are done reading $y$, and what state we're at in the simulation of the DFA $M$. Before doing anything, this NFA needs to simulate starting at the guessed state $q_x$, and then after guessing that we are done reading $y$, the NFA needs to simulate starting at $s$ again. The NFA will accept if and only if the state we end up at is in fact $q_x$.

Thus, given a DFA $M = (Q, \delta, s, A)$ for $L$, we construct an NFA $N = (Q', \delta', s', A')$ for $\textsc{Cycle}(L)$ as follows.

- The ***states*** of $N$ are

$$\boxed{Q' := \big(Q \times Q \times \{\textsc{Before}, \textsc{After}\}\big) \cup \{s'\}}$$

  where $s'$ is a ***new starting state***.

  The meaning of a state $(q, r, b) \in Q \times Q \times \{\textsc{Before}, \textsc{After}\}$ is as follows: $q$ is the state we guessed to be $q_x$, $r$ is the current state in the simulation, and $b$ is $\textsc{Before}$ if we haven't guessed that $y$ is done, and $\textsc{After}$ if we have.

- From the new starting state $s'$, we have $\varepsilon$-transitions to each state of the form $(q, q, \textsc{Before})$. Each of these $\varepsilon$-transitions represents guessing a state $q$ to be $q_x$ and starting the simulation of $M$ at $q$. We need these to be $\varepsilon$-transitions so that we can non-deterministically choose one of them before the simulation.

  For each state $(q, r, \textsc{Before})$ where $r \in A$ ($r$ is an accepting state in $M$), we have an $\varepsilon$-transition to the state $(q, s, \textsc{After})$ ($s$ is the starting state of $M$). This represents the possibility of guessing that we are done reading $y$, and are thus ready to read in $x$ and verify that the initial guess was correct. Even if $M$ only had one accepting state $r$, we would still use an $\varepsilon$-transition instead of combining $(q, r, \text{Before})$ and $(q, s, \text{After})$ to avoid the possibility of being able to move back to Before states from an After state.

  Otherwise, we are just reading in characters and simulating various parts of $M$ on the input. This can be done by advancing $r$ along $\delta$.

  In summary, we can write down the following ***transition function*** $\delta'$:

$$\boxed{\begin{aligned} \delta'(s', \varepsilon) &:= \{(q, q, \textsc{Before}) \mid q \in Q\} \\ \text{For } r \in A, \quad \delta'\big((q, r, \textsc{Before}), \varepsilon\big) &:= \{(q, s, \textsc{After})\} \\ \delta'\big((q, r, b), a\big) &:= \{(q, \delta(r, a), b)\} \end{aligned}}$$

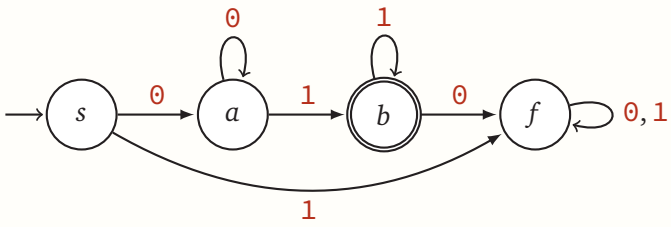
  ***All unspecified transitions do not exist*** (i.e., have value $\varnothing$).

- Finally, all that remains to be specified is the set of ***accepting states*** $A'$. Recall that our guesses were correct if, after reading in what we guessed to be the $x$ part of $w$ from $s$, we end at what we guessed to be $q_x$. In other words:

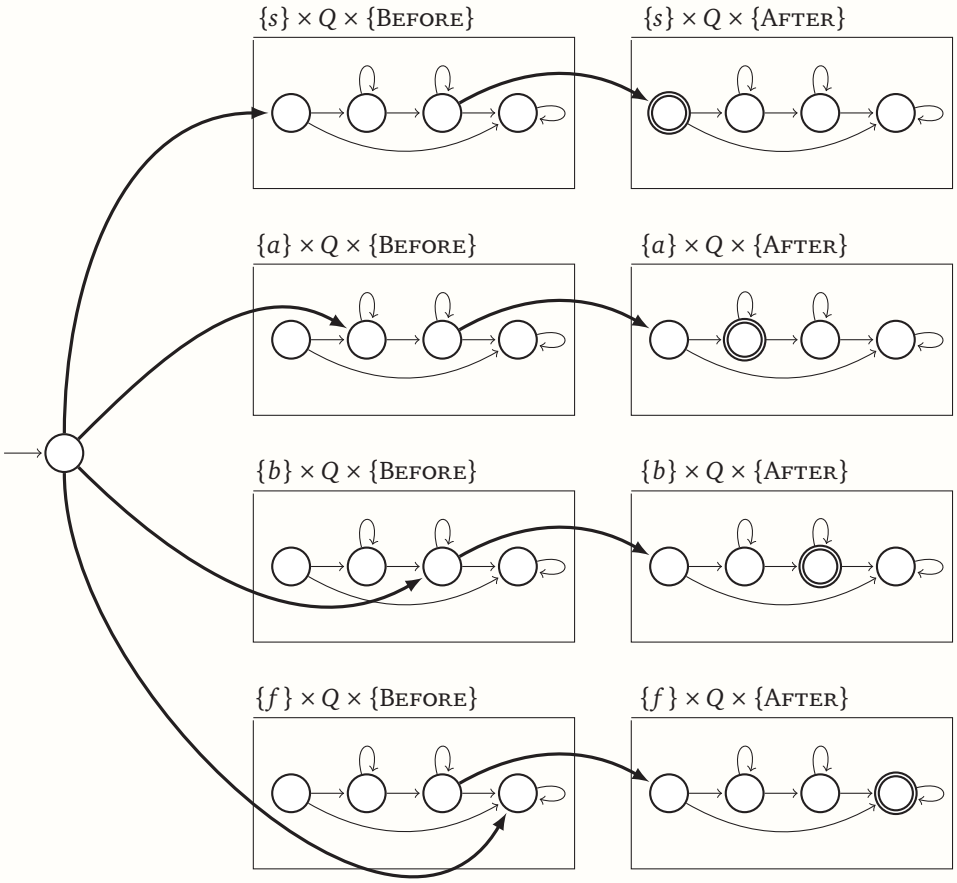
$$\boxed{A' := \{(q, q, \textsc{After}) \mid q \in Q\}}$$

This completes the construction.

As a concrete example, consider the following DFA for the language $L = L(\texttt{00}^*\texttt{11}^*)$:




The NFA we construct for $\textsc{Cycle}(L)$ is shown below. **Bolded** transitions are *new* $\varepsilon$-transitions; all other transitions are the same as their corresponding transitions in the DFA above.



Finally, let's verify that the general construction is correct, i.e., $L(N) = \textsc{Cycle}(L)$. The following "proof" is missing a few details, in particular, a proof of the interaction between $(\delta')^*$ and $\delta^*$. Feel free to follow along on the concrete example on the previous page.

- First, suppose that $w \in L(N)$. Then there is some path that we can follow while reading in $w$ such that, starting from $s'$, we reach an accepting state, i.e., a state of the form $(q, q, \textsc{After})$.

  By construction, this path must start by taking an $\varepsilon$-transition to a state of the form $(q, q, \textsc{Before})$, followed by some *walk* inside of the states $\{q\} \times Q \times \{\textsc{Before}\}$. At some point, an $\varepsilon$-transition leaving $\{q\} \times Q \times \{\textsc{Before}\}$ must be taken; call the input read up to this point $y$. The $\varepsilon$-transition leaving $\{q\} \times Q \times \{\textsc{Before}\}$ must come from a state of the form $(q, r, \textsc{Before})$ where $r \in A$, and goes to $(q, s, \textsc{After})$; at this point reading the rest of the input must lead to $(q, q, \textsc{After})$. Call the rest of the input $x$.

  At this point we know that $\delta^*(s, x) = q$, and furthermore $\delta^*(q, y) \in A$. In other words, $xy \in L$. We therefore conclude that $w = yx \in \textsc{Cycle}(L)$.

  This shows that $L(N) \subseteq \textsc{Cycle}(L)$.

- Next, suppose $w \in \textsc{Cycle}(L)$. Then $w = yx$ where $xy \in L$. We need to show that there is a path from $s'$ to a an accepting state in $N$, i.e., a state of the form $(q, q, \textsc{After})$.

  Set $q = \delta^*(s, x)$. Before reading any of $w$ we take an $\varepsilon$-transition to $(q, q, \textsc{Before})$. After reading $y$, we must be in the state $(q, \delta^*(q, y), \textsc{Before})$ where $\delta^*(q, y) = \delta^*(\delta^*(s, x), y) = \delta^*(s, xy) \in A$. By construction there exists an $\varepsilon$-transition from $(q, \delta^*(q, y), \textsc{Before})$ to $(q, s, \textsc{After})$. After taking this transition, we read in the rest of $w$ (i.e., $x$) and reach $(q, \delta^*(s, x), \textsc{After}) = (q, q, \textsc{After})$, which is an accepting state of $N$.

  We therefore conclude that $w \in L(N)$, and thus $\textsc{Cycle}(L) \subseteq L(N)$.