Describe recursive backtracking algorithms for the following problems. *Don't worry about running times.*

1. Given an array $A[1..n]$ of integers, compute the length of a ***longest increasing subsequence***.

Solution (#1 of ∞): Add a sentinel value $A[0] = -\infty$. Let $LIS(i, j)$ denote the length of the longest increasing subsequence of $A[j..n]$ where every element is larger than $A[i]$. This function obeys the following recurrence:

$$LIS(i,j) = \begin{cases} 0 & \text{if } j > n \\ LIS(i, j+1) & \text{if } j \le n \text{ and } A[i] \ge A[j] \\ \max\{LIS(i, j+1), 1 + LIS(j, j+1)\} & \text{otherwise} \end{cases}$$

We need to compute $LIS(0, 1)$. ■

Solution (#2 of ∞): Add a sentinel value $A[n+1] = -\infty$. Let $LIS(i, j)$ denote the length of the longest increasing subsequence of $A[1..j]$ where every element is smaller than $A[j]$. This function obeys the following recurrence:

$$LIS(i,j) = \begin{cases} 0 & \text{if } i < 1 \\ LIS(i-1, j) & \text{if } i \ge 1 \text{ and } A[i] \ge A[j] \\ \max\{LIS(i-1, j), 1 + LIS(i-1, i)\} & \text{otherwise} \end{cases}$$

We need to compute $LIS(n, n+1)$. ■

Solution (#3 of ∞): Let $LIS(i)$ denote the length of the longest increasing subsequence of $A[i..n]$ that begins with $A[i]$. This function obeys the following recurrence:

$$LIS(i) = \begin{cases} 1 & \text{if } A[j] \le A[i] \text{ for all } j > i \\ 1 + \max\left\{LIS(j) \;\middle|\; j > i \text{ and } A[j] > A[i]\right\} & \text{otherwise} \end{cases}$$

(The first case is actually redundant if we define $\max \varnothing = 0$.) We need to compute $\max_i LIS(i)$. ■

Solution (#4 of ∞): Add a sentinel value $A[0] = -\infty$. Let $LIS(i)$ denote the length of the longest increasing subsequence of $A[i..n]$ that begins with $A[i]$. This function obeys the following recurrence:

$$LIS(i) = \begin{cases} 1 & \text{if } A[j] \le A[i] \text{ for all } j > i \\ 1 + \max\left\{LIS(j) \;\middle|\; j > i \text{ and } A[j] > A[i]\right\} & \text{otherwise} \end{cases}$$

(The first case is actually redundant if we define $\max \varnothing = 0$.) We need to compute $LIS(0) - 1$; the -1 removes the sentinel $-\infty$ from the start of the subsequence. ■

Solution (#5 of ∞): Add sentinel values $A[0] = -\infty$ and $A[n+1] = \infty$. Let $LIS(j)$ denote the length of the longest increasing subsequence of $A[1..j]$ that ends with $A[j]$. This function obeys the following recurrence:

$$LIS(j) = \begin{cases} 1 & \text{if } j = 0 \\ 1 + \max\left\{LIS(i) \;\middle|\; i < j \text{ and } A[i] < A[j]\right\} & \text{otherwise} \end{cases}$$

We need to compute $LIS(n+1) - 2$; the -2 removes the sentinels $-\infty$ and ∞ from the subsequence. ■

2. Given an array $A[1\,..\,n]$ of integers, compute the length of a ***longest decreasing subsequence***.

   Solution (one of many): Add a sentinel value $A[0] = \infty$. Let $LDS(i, j)$ denote the length of the longest decreasing subsequence of $A[j\,..\,n]$ where every element is smaller than $A[i]$. This function obeys the following recurrence:

$$LDS(i,j) = \begin{cases} 0 & \text{if } j > n \\ LDS(i, j+1) & \text{if } j \le n \text{ and } A[i] \le A[j] \\ \max\left\{LDS(i, j+1), 1 + LIS(j, j+1)\right\} & \text{otherwise} \end{cases}$$

   We need to compute $LDS(0, 1)$. ■

   Solution (clever): Multiply every element of A by -1, and then compute the length of the longest increasing subsequence using the algorithm from problem 1. ■

3. Given an array $A[1..n]$ of integers, compute the length of a ***longest alternating subsequence***.

Solution (one of many): We define two functions:

- Let $LAS^+(i, j)$ denote the length of the longest alternating subsequence of $A[j..n]$ whose first element (if any) is larger than $A[i]$ and whose second element (if any) is smaller than its first.
- Let $LAS^-(i, j)$ denote the length of the longest alternating subsequence of $A[j..n]$ whose first element (if any) is smaller than $A[i]$ and whose second element (if any) is larger than its first.

These two functions satisfy the following mutual recurrences:

$$LAS^+(i,j) = \begin{cases} 0 & \text{if } j > n \\ LAS^+(i, j+1) & \text{if } j \le n \text{ and } A[j] \le A[i] \\ \max\left\{LAS^+(i, j+1), 1 + LAS^-(j, j+1)\right\} & \text{otherwise} \end{cases}$$

$$LAS^-(i,j) = \begin{cases} 0 & \text{if } j > n \\ LAS^-(i, j+1) & \text{if } j \le n \text{ and } A[j] \ge A[i] \\ \max\left\{LAS^-(i, j+1), 1 + LAS^+(j, j+1)\right\} & \text{otherwise} \end{cases}$$

To simplify computation, we consider two different sentinel values $A[0]$. First we set $A[0] = -\infty$ and let $\ell^+ = LAS^+(0, 1)$. Then we set $A[0] = +\infty$ and let $\ell^- = LAS^-(0, 1)$. Finally, the length of the longest alternating subsequence of A is $\max\{\ell^+, \ell^-\}$. ■

Solution (one of many): We define two functions:

- Let $LAS^+(i)$ denote the length of the longest alternating subsequence of $A[i..n]$ that starts with $A[i]$ and whose second element (if any) is larger than $A[i]$.
- Let $LAS^-(i)$ denote the length of the longest alternating subsequence of $A[i..n]$ that starts with $A[i]$ and whose second element (if any) is smaller than $A[i]$.

These two functions satisfy the following mutual recurrences:

$$LAS^+(i) = \begin{cases} 1 & \text{if } A[j] \le A[i] \text{ for all } j > i \\ 1 + \max\left\{LAS^-(j) \;\middle|\; j > i \text{ and } A[j] > A[i]\right\} & \text{otherwise} \end{cases}$$

$$LAS^-(i) = \begin{cases} 1 & \text{if } A[j] \ge A[i] \text{ for all } j > i \\ 1 + \max\left\{LAS^+(j) \;\middle|\; j > i \text{ and } A[j] < A[i]\right\} & \text{otherwise} \end{cases}$$

We need to compute $\max_i \max\{LAS^+(i), LAS^-(i)\}$. ■

To think about later:

4. Given an array $A[1..n]$ of integers, compute the length of a longest ***convex*** subsequence of A.

Solution: Let $LCS(i,j)$ denote the length of the longest convex subsequence of $A[i..n]$ whose first two elements are $A[i]$ and $A[j]$. This function obeys the following recurrence:

$$LCS(i,j) = 1 + \max\left\{LCS(j,k) \;\middle|\; j < k \leq n \text{ and } A[i] + A[k] > 2A[j]\right\}$$

Here we define $\max\varnothing = 0$; this gives us a working base case. The length of the longest convex subsequence is $\max_{1\leq i<j\leq n} LCS(i,j)$. ■

Solution (with sentinels): Assume without loss of generality that $A[i] \geq 0$ for all i. (Otherwise, we can add $|m|$ to each $A[i]$, where m is the smallest element of $A[1..n]$.) Add two sentinel values $A[0] = 2M + 1$ and $A[-1] = 4M + 3$, where M is the largest element of $A[1..n]$.

Let $LCS(i,j)$ denote the length of the longest convex subsequence of $A[i..n]$ whose first two elements are $A[i]$ and $A[j]$. This function obeys the following recurrence:

$$LCS(i,j) = 1 + \max\left\{LCS(j,k) \;\middle|\; j < k \leq n \text{ and } A[i] + A[k] > 2A[j]\right\}$$

Here we define $\max\varnothing = 0$; this gives us a working base case.

Finally, we claim that the length of the longest convex subsequence of $A[1..n]$ is $LCS(-1,0)-2$.

Proof: First, consider any convex subsequence S of $A[1..n]$, and suppose its first element is $A[i]$. Then we have $A[-1] - 2A[0] + A[i] = 4M + 3 - 2(2M + 1) + A[i] = A[i] + 1 > 0$, which implies that $A[-1] \cdot A[0] \cdot S$ is a convex subsequence of $A[-1..n]$. So the longest convex subsequence of $A[1..n]$ has length at most $LCS(-1,0) - 2$.

On the other hand, removing $A[-1]$ and $A[0]$ from any convex subsequence of $A[-1..n]$ laves a convex subsequence of $A[1..n]$. So the longest subsequence of $A[1..n]$ has length at least $LCS(-1,0) - 2$. □

■

5. Given an array $A[1..n]$, compute the length of a longest ***palindrome*** subsequence of A.

Solution (naïve): Let $LPS(i,j)$ denote the length of the longest palindrome subsequence of $A[i..j]$. This function obeys the following recurrence:

$$LPS(i,j) = \begin{cases} 0 & \text{if } i > j \\ 1 & \text{if } i = j \\ \max\left\{\begin{array}{c} LPS(i+1,j) \\ LPS(i,j-1) \end{array}\right\} & \text{if } i < j \text{ and } A[i] \neq A[j] \\ \max\left\{\begin{array}{c} 2 + LPS(i+1,j-1) \\ LPS(i+1,j) \\ LPS(i,j-1) \end{array}\right\} & \text{otherwise} \end{cases}$$

We need to compute $LPS(1,n)$. ■

Solution (with greedy optimization): Let $LPS(i,j)$ denote the length of the longest palindrome subsequence of $A[i..j]$. Before stating a recurrence for this function, we make the following useful observation.[1]

Claim 1. *If $i < j$ and $A[i] = A[j]$, then $LPS(i,j) = 2 + LPS(i+1,j-1)$.*

Proof: Suppose $i < j$ and $A[i] = A[j]$. Fix an arbitrary longest palindrome subsequence S of $A[i..j]$. There are four cases to consider.

- If S uses neither $A[i]$ nor $A[j]$, then $A[i] \bullet S \bullet A[j]$ is a palindrome subsequence of $A[i..j]$ that is longer than S, which is impossible.
- Suppose S uses $A[i]$ but not $A[j]$. Let $A[k]$ be the last element of S. If $k = i$, then $A[i] \bullet A[j]$ is a palindrome subsequence of $A[i..j]$ that is longer than S, which is impossible. Otherwise, replacing $A[k]$ with $A[j]$ gives us a palindrome subsequence of $A[i..j]$ with the same length as S that uses both $A[i]$ and $A[j]$.
- Suppose S uses $A[j]$ but not $A[i]$. Let $A[h]$ be the first element of S. If $h = j$, then $A[i] \bullet A[j]$ is a palindrome subsequence of $A[i..j]$ that is longer than S, which is impossible. Otherwise, replacing $A[h]$ with $A[i]$ gives us a palindrome subsequence of $A[i..j]$ with the same length as S that uses both $A[i]$ and $A[j]$.
- Finally, S might include both $A[i]$ and $A[j]$.

In all cases, we find either a contradiction or a longest palindrome subsequence of $A[i..j]$ that uses both $A[i]$ and $A[j]$. □

Claim 1 implies that the function LPS satisfies the following recurrence:

$$LPS(i,j) = \begin{cases} 0 & \text{if } i > j \\ 1 & \text{if } i = j \\ \max\left\{LPS(i+1,j),\ LPS(i,j-1)\right\} & \text{if } i < j \text{ and } A[i] \neq A[j] \\ 2 + LPS(i+1,j-1) & \text{otherwise} \end{cases}$$

We need to compute $LPS(1,n)$. ■

[1] And yes, optimizations like this require a proof of correctness, both in homework and on exams. Premature optimization is the root of all evil.