Prove that each of the following languages is \textbf{not} regular.

1. \{0^{2n}1^n \mid n \geq 0\}

\textbf{Solution (verbose):} Let \( F \) be the language \( 0^* \).

Let \( x \) and \( y \) be arbitrary strings in \( F \).

Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).

Let \( z = 0^i1^i \).

Then \( xz = 0^{2i}1^i \in L \).

And \( yz = 0^{i+j}1^i \notin L \), because \( i + j \neq 2i \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular. \hfill \blacksquare

\textbf{Solution (concise):} For all non-negative integers \( i \neq j \), the strings \( 0^i \) and \( 0^j \) are distinguished by the suffix \( 0^i1^i \), because \( 0^{2i}1^i \in L \) but \( 0^{i+j}1^i \notin L \). Thus, the language \( 0^* \) is an infinite fooling set for \( L \). \hfill \blacksquare

\textbf{Solution (concise, different fooling set):} For all non-negative integers \( i \neq j \), the strings \( 0^{2i} \) and \( 0^{2j} \) are distinguished by the suffix \( 1^i \), because \( 0^{2i}1^i \in L \) but \( 0^{2i+1}1^j \notin L \). Thus, the language \( (00)^* \) is an infinite fooling set for \( L \). \hfill \blacksquare
2. \{\theta^m \Gamma^n | m \neq 2n\}

Solution (verbose): Let \( F \) be the language \( \theta^* \).

Let \( x \) and \( y \) be arbitrary strings in \( F \).

Then \( x = \theta^i \) and \( y = \theta^j \) for some non-negative integers \( i \neq j \).

Let \( z = \theta^i \Gamma^i \).

Then \( xz = \theta^{2i} \Gamma^i \notin L \).

And \( yz = \theta^{i+j} \Gamma^i \notin L \), because \( i + j \neq 2i \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

Solution (concise, different fooling set): For all non-negative integers \( i \neq j \), the strings \( \theta^{2i} \) and \( \theta^{2j} \) are distinguished by the suffix \( \Gamma^i \), because \( \theta^{2i} \Gamma^i \notin L \) but \( \theta^{2j} \Gamma^i \in L \). Thus, the language \( (\theta\theta)^* \) is an infinite fooling set for \( L \).

3. \{\theta^{2n} | n \geq 0\}

Solution (verbose): Let \( F = L = \{\theta^{2n} | n \geq 0\} \).

Let \( x \) and \( y \) be arbitrary elements of \( F \).

Then \( x = \theta^2x \) and \( y = \theta^2y \) for some non-negative integers \( x \) and \( y \).

Let \( z = \theta^2 \).

Then \( xz = \theta^2 \theta^2 = \theta^{2x+1} \in L \).

And \( yz = \theta^2 \theta^2 = \theta^{2x+2} \notin L \), because \( i \neq j \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

Solution (concise): For any non-negative integers \( i \neq j \), the strings \( \theta^{2i} \) and \( \theta^{2j} \) are distinguished by the suffix \( \theta^2 \), because \( \theta^{2i} \theta^{2i} = \theta^{2i+1} \in L \) but \( \theta^{2j} \theta^{2i} = \theta^{2i+2} \notin L \). Thus \( L \) itself is an infinite fooling set for \( L \).

4. Strings over \{0,1\} where the number of 0s is exactly twice the number of 1s.

Solution (verbose): Let \( F \) be the language \( \theta^* \).

Let \( x \) and \( y \) be arbitrary strings in \( F \).

Then \( x = \theta^i \) and \( y = \theta^j \) for some non-negative integers \( i \neq j \).

Let \( z = \theta^i \Gamma^i \).

Then \( xz = \theta^{2i} \Gamma^i \in L \).

And \( yz = \theta^{i+j} \Gamma^i \notin L \), because \( i + j \neq 2i \).

Thus, \( F \) is a fooling set for \( L \).

Because \( F \) is infinite, \( L \) cannot be regular.

Solution (concise): For any non-negative integers \( i \neq j \), the strings \( \theta^{2i} \) and \( \theta^{2j} \) are distinguished by the suffix \( \Gamma^i \), because \( \theta^{2i} \theta^{2i} = \theta^{2i+1} \in L \) but \( \theta^{2j} \theta^{2i} = \theta^{2i+2} \notin L \). Thus \( L \) itself is an infinite fooling set for \( L \).
Solution (concise, different fooling set): For all non-negative integers \( i \neq j \), the strings \( 0^{2i} \) and \( 0^{2j} \) are distinguished by the suffix \( 1^i \), because \( 0^{2i}1^i \in L \) but \( 0^{2j}1^i \notin L \). Thus, the language \((00)^*\) is an infinite fooling set for \( L \). 

Solution (closure properties): If \( L \) were regular, then the language

\[
 L \cap 0^*1^* = \{0^{2n}1^n \mid n \geq 0\}
\]

would also be regular since regular languages are closed under intersection but we have seen in Problem 1 that \( \{0^{2n}1^n \mid n \geq 0\} \) is not regular.

Another solution based on closure properties. If \( L \) were regular, then the language

\[
(0 + 1)^* \setminus L \cap 0^*1^* = \{0^m1^n \mid m \neq 2n\}
\]

would also be regular, because regular languages are closed under complement and intersection. But we just proved that \( \{0^m1^n \mid m \neq 2n\} \) is not regular in problem 2. \([Yes, this proof would be worth full credit, either in homework or on an exam.\)]

Note that the proofs based on closure properties relied on non-regularity of some previously known languages. One could also think of the proofs as allowing you to simplify the initial language to a more structured one which may be easier to work with.
5. Strings of properly nested parentheses \((\)), brackets \([\)[, and braces \({}\). For example, the string \((\{\})\) is in this language, but the string \((\})\) is not, because the left and right delimiters don’t match.

**Solution (verbose):** Let \(F\) be the language \(\{^*\}^*\).

Let \(x\) and \(y\) be arbitrary strings in \(F\).

Then \(x = \{^i\) and \(y = \{^j\) for some non-negative integers \(i \neq j\).

Let \(z = \{^i \).

Then \(xz = \{^i \}^i \in L\).

And \(yz = \{^i \}^j \not\in L\), because \(i \neq j\).

Thus, \(F\) is a fooling set for \(L\).

Because \(F\) is infinite, \(L\) cannot be regular.

**Solution (concise):** For any non-negative integers \(i \neq j\), the strings \(\{^i\) and \(\{^j\) are distinguished by the suffix \(\}^i\), because \(\{^i \}^i \in L\) but \(\{^i \}^j \not\in L\). Thus, the language \(\{^*\}^*\) is an infinite fooling set.

**Solution (closure properties):** If \(L\) were regular, then the language \(L \cap \{^*\}^* = \{(\{^n\) \mid n \geq 0\}\) would be regular. The language \(\{(\{^n\) \mid n \geq 0\}\) is the same as \(\{0^n1^n \mid n \geq 0\}\) modulo changing the symbol names and is not regular from lecture. Thus \(L\) is not regular.

6. Strings of the form \(w_1\#w_2\#\cdots\#w_n\) for some \(n \geq 2\), where each substring \(w_i\) is a string in \(\{0, 1\}^*\), and some pair of substrings \(w_i\) and \(w_j\) are equal.

**Solution (verbose):** Let \(F\) be the language \(\{0\}^*\).

Let \(x\) and \(y\) be arbitrary strings in \(F\).

Then \(x = \{^i\) and \(y = \{^j\) for some non-negative integers \(i \neq j\).

Let \(z = \#\{^i\).

Then \(xz = \{^i \}^i \in L\).

And \(yz = \{^i \}^j \not\in L\), because \(i \neq j\).

Thus, \(F\) is a fooling set for \(L\).

Because \(F\) is infinite, \(L\) cannot be regular.

**Solution (concise):** For any non-negative integers \(i \neq j\), the strings \(\{^i\) and \(\{^j\) are distinguished by the suffix \(#\{^i\), because \(\{^i \}^i \in L\) but \(\{^j \}^i \not\in L\). Thus, the language \(\{0\}^*\) is an infinite fooling set.
Work on these later:

7. \( \{ \theta^{n^2} \mid n \geq 0 \} \)

**Solution:** Let \( x \) and \( y \) be distinct arbitrary strings in \( L \).

Without loss of generality, \( x = \theta^{i^2} \) and \( y = \theta^{j^2} \) for some \( i > j \geq 0 \).

Let \( z = \theta^{2i+1} \).

Then \( xz = \theta^{i^2+2i+1} = \theta^{(i+1)^2} \in L \).

On the other hand, \( yz = \theta^{i^2+2j+1} \not\in L \), because \( i^2 < i^2 + 2j + 1 < (i + 1)^2 \).

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( L \) is an infinite fooling set for \( L \), so \( L \) cannot be regular. ■

**Solution:** Let \( x \) and \( y \) be distinct arbitrary strings in \( \theta^* \).

Without loss of generality, \( x = \theta^i \) and \( y = \theta^j \) for some \( i > j \geq 0 \).

Let \( z = \theta^{i^2+i+1} \).

Then \( xz = \theta^{i^2+2i+1} = \theta^{(i+1)^2} \in L \).

On the other hand, \( yz = \theta^{i^2+i+j+1} \not\in L \), because \( i^2 < i^2 + i + j + 1 < (i + 1)^2 \).

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( \theta^* \) is an infinite fooling set for \( L \), so \( L \) cannot be regular. ■

**Solution:** Let \( x \) and \( y \) be distinct arbitrary strings in \( \theta^{0000}^* \).

Without loss of generality, \( x = \theta^i \) and \( y = \theta^j \) for some \( i > j \geq 3 \).

Let \( z = \theta^{i^2-i} \).

Then \( xz = \theta^{i^2} \in L \).

On the other hand, \( yz = \theta^{i^2-i+j} \not\in L \), because

\[ (i - 1)^2 = i^2 - 2i + 1 < i^2 - i < i^2 - i + j < i^2. \]

(The first inequalities requires \( i \geq 2 \), and the second \( j \geq 1 \).)

Thus, \( z \) distinguishes \( x \) and \( y \).

We conclude that \( \theta^{0000}^* \) is an infinite fooling set for \( L \), so \( L \) cannot be regular. ■
8. \( \{ w \in (0+1)^* \mid w \text{ is the binary representation of a perfect square} \} \)

**Solution:** We design our fooling set around numbers of the form \((2^k+1)^2 = 2^{2k} + 2^{k+1} + 1 = 10^{k-2}10^k1 \in L\), for any integer \( k \geq 2 \). The argument is somewhat simpler if we further restrict \( k \) to be even.

Let \( F = 1(00)^*1 \), and let \( x \) and \( y \) be arbitrary strings in \( F \).

Then \( x = 10^{2i-2}1 \) and \( y = 10^{2j-2}1 \), for some positive integers \( i \neq j \).

Without loss of generality, assume \( i < j \). (Otherwise, swap \( x \) and \( y \).)

Let \( z = 0^{2i}1 \).

Then \( xz = 10^{2i-2}10^{2i}1 \) is the binary representation of \( 2^{4i} + 2^{2i+1} + 1 = (2^{2i} + 1)^2 \), and therefore \( xz \in L \).

On the other hand, \( yz = 10^{2j-2}10^{2i}1 \) is the binary representation of \( 2^{2i+2j} + 2^{2i+1} + 1 \). Simple algebra gives us the inequalities

\[
(2^{i+j})^2 = 2^{2i+2j} < 2^{2i+2j} + 2^{2i+1} + 1 < 2^{2(i+j)} + 2^{i+j+1} + 1 = (2^{i+j} + 1)^2.
\]

So \( 2^{2i+2j} + 2^{2i+1} + 1 \) lies between two consecutive perfect squares, and thus is not a perfect square, which implies that \( yz \notin L \).

We conclude that \( F \) is a fooling set for \( L \). Because \( F \) is infinite, \( L \) cannot be regular. ■