1. Given a directed graph $G = (V, E)$ with non-negative edge lengths $\ell(e), e \in E$ and a node $s \in V$, describe an algorithm to find the length of a shortest cycle containing the node s.

2. Suppose we have a collection of cities and different airlines offer flights between various pairs of cities. Some airlines only fly between some pairs of cities. Some pairs of cities are served by many airlines. Each airline charges perhaps different amounts for their one-way tickets.

   - Suppose you'd like to get from City A to City B at the least total cost. Describe an efficient solution. (Your solution may change planes to a different airline as needed.)
   - It turns out that airports charge usage taxes. Different airports may charge different amounts in tax. Your cost of traveling from A to B now includes all of the flight costs, plus all of the taxes of the airports that you stopover along the way from A to B. Model this as a graph problem and give an efficient solution to find the least cost way to get from A to B.

3. There are n galaxies connected by m intergalactic teleport-ways. Each teleport-way joins two galaxies and can be traversed in both directions. However, the company that runs the teleport-ways has established an extremely lucrative cost structure: Anyone can teleport *further* from their home galaxy at no cost whatsoever, but teleporting *toward* their home galaxy is prohibitively expensive.

   Judy has decided to take a sabbatical tour of the universe by visiting as many galaxies as possible, starting at her home galaxy. To save on travel expenses, she wants to teleport away from her home galaxy at every step, except for the very last teleport home.

   Describe and analyze an algorithm to compute the maximum number of galaxies that Judy can visit. Your input consists of an undirected graph G with n vertices and m edges describing the teleport-way network, an integer $1 \leq s \leq n$ identifying Judy's home galaxy, and an array $D[1..n]$ containing the distances of each galaxy from s.

To think about later: You probably heard of the phrase "six degrees of separation" and the "small world" phenomenon; see https://en.wikipedia.org/wiki/Six_degrees_of_separation. The idea is that in many interesting networks people or objects are within a small distance of each other. At the same time we believe in "locality" in that each person may only a small number of people compared to the total population. The next two problems explore the tradeoffs between diameter and degree in a graph to explore this in a more quantitative fashion.

4. Suppose G is a graph with maximum degree d. The diameter of the graph is $\max_{u,v} \text{dist}(u, v)$. Prove that the diameter of the graph is $\Omega(\log_d n)$ where n is the number of nodes. It is easier to consider $d = 5$ or some other small constant for simplicity. *Hint:* Consider the BFS layers starting at any vertex v.

   The point of the problem is to show that if all degrees are small then the diameter must grow with the number of nodes.

5. Suppose the diameter of an undirected simple graph is d. Prove that there is a node with degree at most $3n/d$. *Hint:* Consider the BFS layers for the pair defining the diameter. It is easier to prove a bound such as $9n/d$.

   This problem is to show you that if the diameter is small then there must be a large degree node.