Turing Machines & Computability

Course Trajectory

- Seen lots of algorithms, what *can* be done.
- Interested in *limitations* (algorithmic; time)
- Need more precise definitions of
  - what is a computer / computation
  - what does “polynomial time” mean
- *Turing Machines* useful in answering these questions

"Most General" computer?

- DFAs are simple model of computation.
  - Accept only the regular languages.
- Is there a kind of computer that can accept *any* language, or compute *any* function?
- Recall counting argument:
  - $\{ f \mid f : N \rightarrow \{0,1\} \}$ (just the set of boolean functions)
    - (a) countably infinite
    - (b) uncountably infinite
  - {P : P is a finite length computer program} is
    - (a) countably infinite
    - (b) uncountably infinite

Most General Computer

- If not all functions are computable, *which are?*
- Is there a “most general” model of computer?
- What languages can they recognize?

David Hilbert

- Early 1900s – crisis in math foundations
  - – attempts to formalize resulted in paradoxes, etc.
- 1920, Hilbert's Program:
  "mechanize" mathematics
- Finite axioms, inference rules
  turn crank, determine truth
  needed: axioms *consistent* & *complete*

This slide recycled from Lecture 1

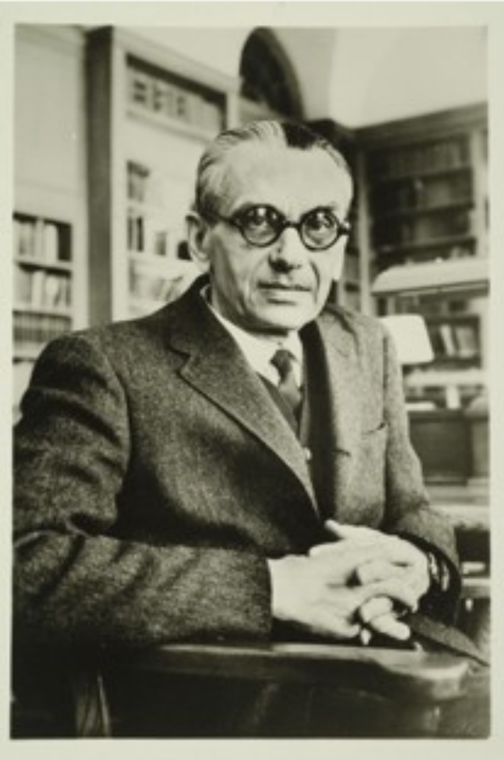
Kurt Gödel

- German logician, at age 25 (1931) proved:

"There are true statements that can't be proved"

  (i.e., "no" to Hilbert)

- Shook the foundations of
  - mathematics
  - philosophy
  - science
  - everything



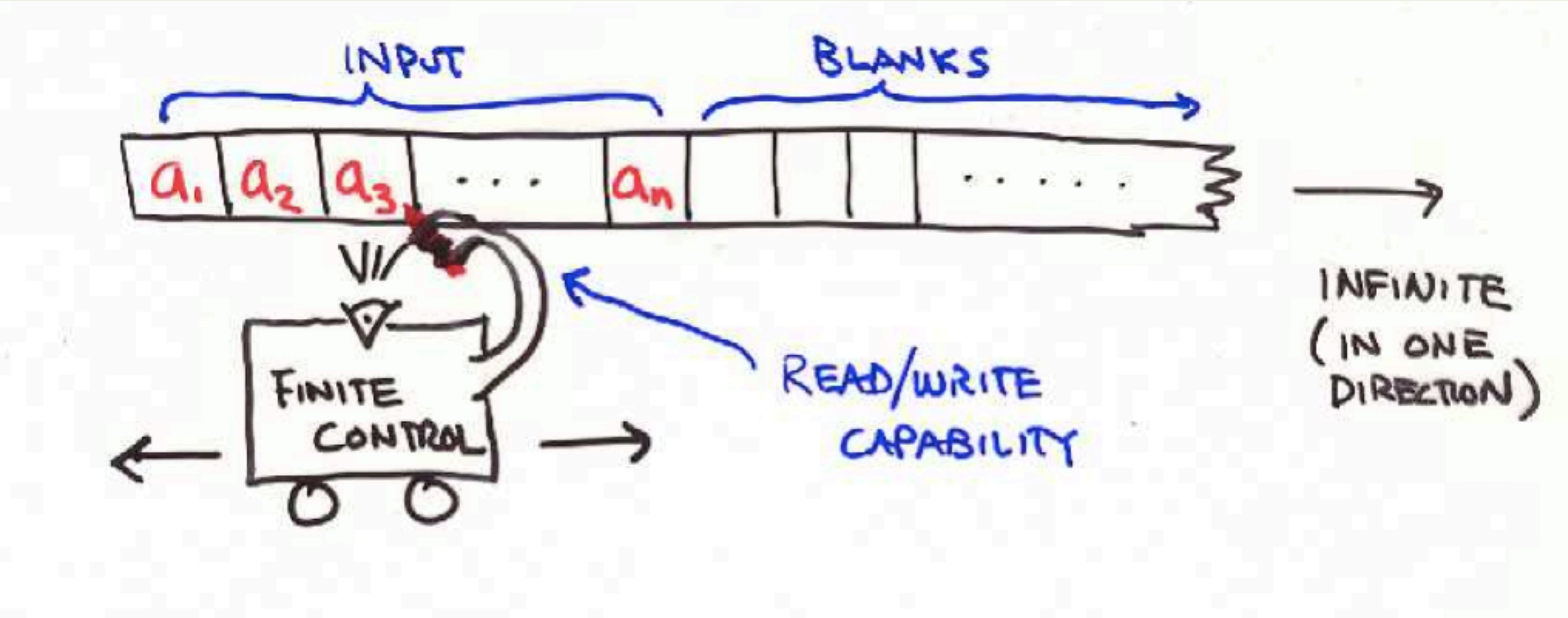
This slide recycled from Lecture 1

Alan Turing

- British mathematician
  - cryptanalysis during WWII
  - arguably, father of AI, Theory
  - several books, movies
- Defined "*computer*", "*program*"

and (1936) provided foundations for investigating fundamental question of what is computable, what is not computable.

This slide recycled from Lecture 1



- DFA with (infinite) tape.
- One move: read, write, move, change state.

Formal Definition

$M = (Q, \Sigma, \Gamma, \delta, q_0, B, q_{accept}, q_{reject})$, where:

- Q is a finite set of states
- Σ is a finite input alphabet
- δ as defined on next page
- Γ is a finite tape alphabet. (Σ a subset of Γ)
- q_0 is the initial state (in Q)
- B in $\Gamma - \Sigma$ is the blank symbol
- q_{accept}, q_{reject} are unique accept, reject states in Q

Transition Function

$$\delta: Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R, S\}$$

current state	symbol scanned	new state	symbol to write	direction to move on tape

$$\delta(q,a) = (p,\ b,\ L)$$

from state q, on reading a:

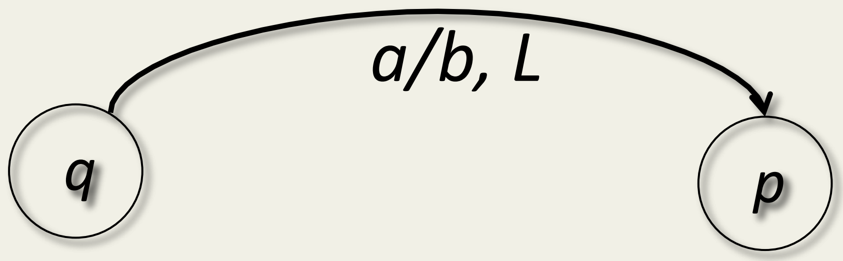
go to state p

write b

move head Left

Graphical Representation

$\delta(q,a) = (p, b, L)$

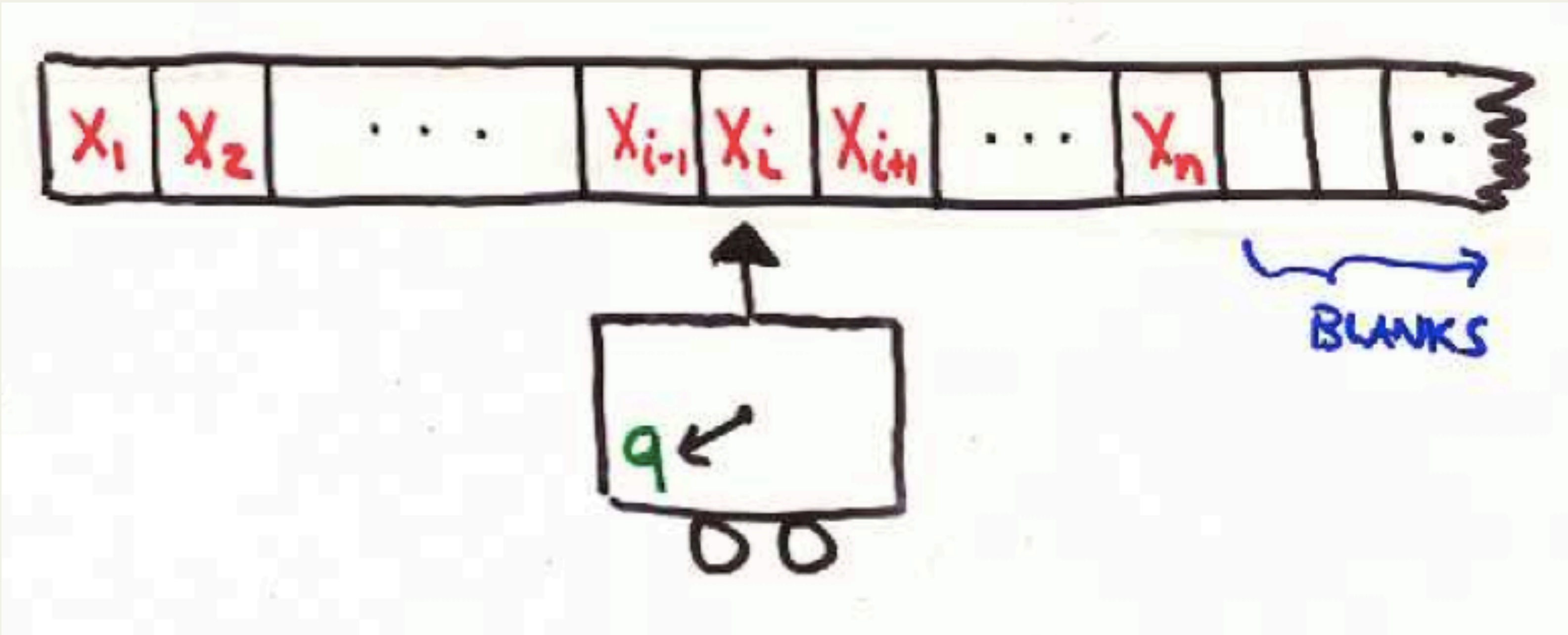


Note: we allow $\delta(q,a)$ to be undefined for some choices of q, a (in which case, M "crashes")

ID: Instantaneous Description

- Contains all necessary information to capture "state of the computation"
- Includes
  - state q of M
  - location of read/write head
  - contents of tape from left edge to rightmost nonblank (or to head, whichever is rightmost)

ID: Instantaneous Description



ID: $X_1X_2\ldots X_{i-1}\ q\ X_iX_{i+1}\ldots X_n$ (q in Q, X_i in Γ)

Relation "→" on IDs

If $\delta(q,X_i) = (p, Y, L)$, then

$$\underbrace{X_1X_2\ldots X_{i-1}\, q\, X_iX_{i+1}\ldots X_n}_{\text{current ID}} \rightarrow \underbrace{X_1X_2\ldots X_{i-2}\, p\, X_{i-1}\, Y\, X_{i+1}}_{\text{next ID}}$$

If $\delta(q,X_i)$ is undefined, then there is no next ID

If M tries to move off left edge, there is no next ID

(in both cases, the machine "crashes")

Capturing many moves...

Define $\rightarrow^*$ as the reflexive, transitive closure of $\rightarrow$

Thus, $ID_1 \rightarrow^* ID_2$ iff M, when run from ID_1, necessarily reaches ID_2 after some finite number of moves.

Initial ID: $q_0 w$ (more often, assume ... $\$q_0 w$)

Accepting ID: $\alpha_1\, q_{accept}\, \alpha_2$ for any α_1, α_2 in Γ^*

(reaches the accepting state with any random junk left on the tape)

Definition of Acceptance

M accepts *w* iff for some α_1, α_2 in Γ^*,

$$q_0 w \rightarrow^* \alpha_1 \, q_{accept} \, \alpha_2$$

M accepts if at *any* time it enters the accept state

Regardless of whether or not

it has scanned all of the input

it has moved back and forth many times

it has completely erased or replaced *w* on the tape

$$L(M) = \{w \mid M \text{ accepts } w\}$$

Non-accepting computation

M doesn't accept w if any of the following occur:

- M enters q_{reject}
- M moves off left edge of tape
- M has no applicable next transition
- M continues computing forever

If M accepts – we can tell: it enters q_{accept}

If M doesn't accept – we may not be able to tell

(c.f. "Halting problem" – later)

“Recursive” vs “Recursively Enumerable”

- *Recursively Enumerable (r.e.) Languages:*

  = {L | there is a TM M such that L(M) = L}

- *Recursive Languages* *(also called “decidable”)*

  = {L | there is a TM M that halts for all w in Σ^*

  and such that L(M) = L }

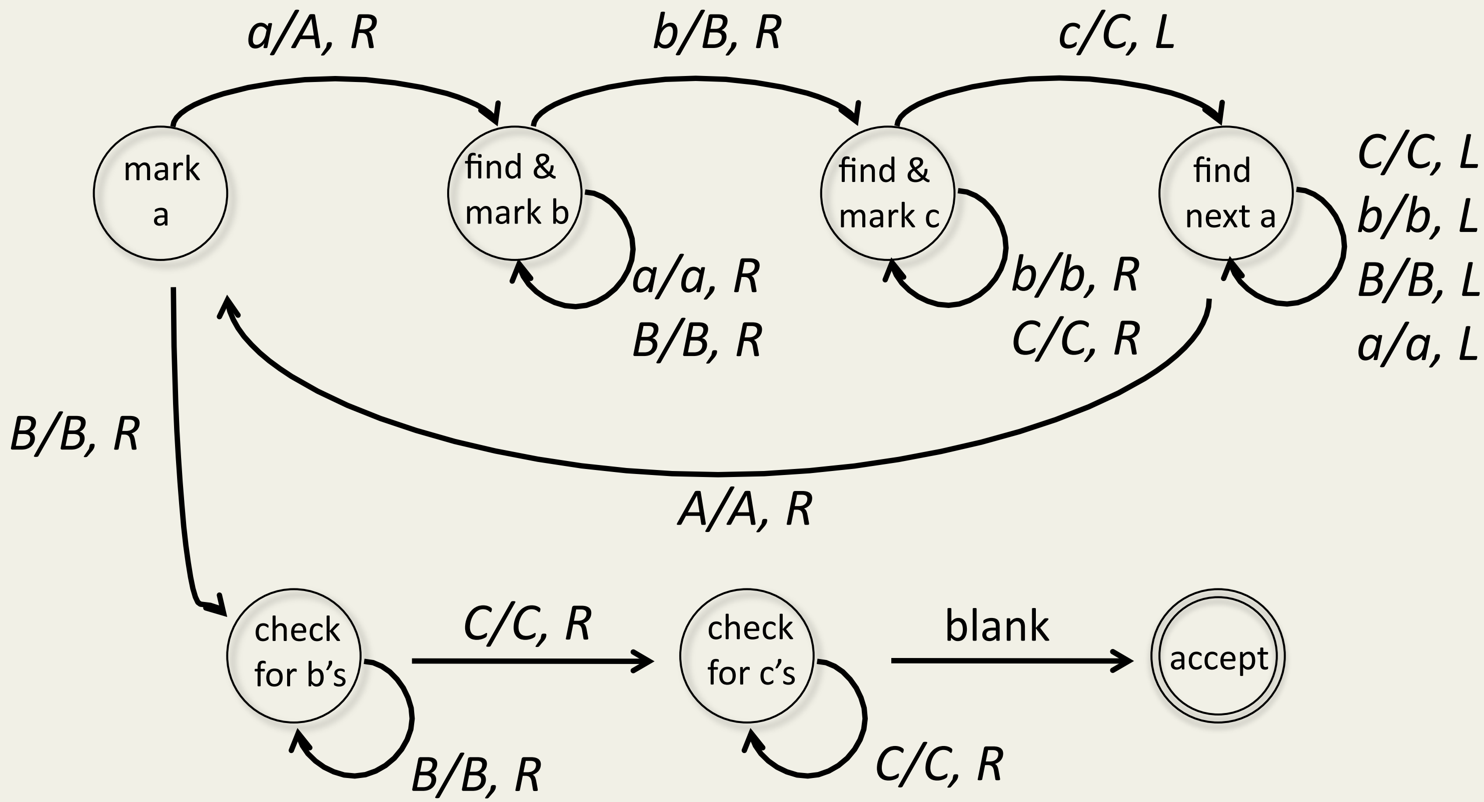
Recursive languages: nice; run *M* on *w* and it will eventually enter either q_{accept} or q_{reject}

r.e. languages: not so nice; can know if *w* in *L*, but not necessarily if *w* is not in *L.*

Fundamental Questions

- Which languages are r.e.?
- Which are recursive?
- What is the difference?
- What properties make a language decidable?

Machine accepting $\{a^n b^n c^n \mid n \geq 1\}$



(This technique is known as "checking off symbols")

Machine to add two n-bit numbers

("high-level" description)

- Assume input is $\$a_1a_2\ldots a_n\#b_1b_2\ldots b_n$
- Pre-process phase
  - sweep right, replacing 0 with 0' and 1 with 1'
- Main loop:
  - erase last bit b_i, and remember it
  - move left to corresponding bit a_i
  - add the bits, plus carry, overwrite a_i with answer
  - remember carry, move right to next (last) bit b_{i-1}

Program Trace (some missing steps)

$10011#11001

$1’0’0’1’1’#1’1’0’0’1’_

$1’0’0’1’1’#1’1’0’0’1’ *b = 1* *c = 0*

$1’0’0’1’1’#1’1’0’0’

$1’0’0’1’1’#1’1’0’0’

$1’0’0’1’1’#1’1’0’0’

$1’0’0’1’1’#1’1’0’0’

$1’0’0’1’1’#1’1’0’0’

$1’0’0’1’1’#1’1’0’0’

$1’0’0’1’0#1’1’0’0’ *c = 1*

$1’0’0’1’0#1’1’0’0’

$1’0’0’1’0#1’1’0’0’

$1’0’0’1’0#1’1’0’0’

$1’0’0’1’0#1’1’0’0’_

$1’0’0’1’0#1’1’0’0’

$1’0’0’1’0#1’1’0’ *b = 0* *c = 1*

$1’0’0’1’0#1’1’0’

$1’0’0’1’0#1’1’0’

$1’0’0’1’0#1’1’0’

$1’0’0’00#1’1’0’ *c = 1*

$1’0’0’00#1’1’0’ etc

Computing Functions with TMs

- number n represented in unary by 0^n

  (well, really 0^{n+1} so we can represent 0...)

- $M(n)$ definition: "output of M on input n":

  IF $q_0 0^n \to^* q_{halt} 0^m$ then $M(n) = m$.

- Every TM M computes some function

  $f_M : \mathrm{N} \to \mathrm{N} \cup \{\text{undefined}\}$.

- Functions with multiple inputs and outputs:

  $M(x,y,z) = (r,s)$ means $q_0 0^x \# 0^y \# 0^z \to^* q_{halt} 0^r \# 0^s$

“Easily” Computed Functions

- addition
- multiplication
- subtraction (max {0, x-y})
- any function you have an “algorithm” for...

Computable Functions

- A (partial) function f: N → N U {undefined} is said to be *computable* iff for some TM M,

  for all x in N, $f(x) = M(x)$ when $f(x)$ is defined

  "f is a (partial) recursive function)"

- A function f: N → N is a *total recursive function* iff for all x in N, $f(x) = M(x)$ for every x.

- If M computes a partial recursive function, it may not halt on some inputs. If M computes a total recursive function, it must halt for all inputs.

Why only f: N → N ?

Q: What about negatives, rationals, reals?

A: We can encode anything as a natural number.

- -5: 100000
- *p/q:* 0^p110^q, *(p/q) + (a/b) as* $0^{pb+aq}110^{bq}$
- reals to given precision, or symbolically
- Intuition: *ANY* function can be coded as a function from N to N

Some TM programming tricks

- checking off symbols
- shifting over
- using finite control memory
- subroutine calls

"Extensions" of TMs

- 2-way infinite tape
- multiple tracks
- multiple tapes
- multi-dimensional TMs
- nondeterministic TMs
- --- bells & whistles

Goal:

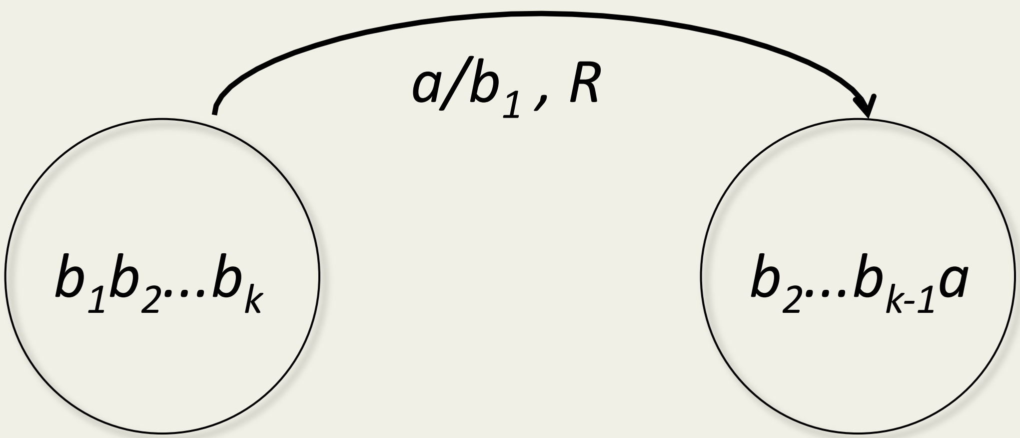
Convince you of the power of the basic model

Checking off symbols

- Use additional tape symbols to represent a "checked-off" character.
- E.g., for each symbol a in Γ, also include "$a_{\checkmark}$" to represent a marked a.
- We essentially did this using A for checked a, or 0' for checked 0, in our previous examples.

Shifting over

- Sometimes need extra cells
- Can shift-over by any number of cells
  - Shift-by-k: Use states to remember previous $|\Gamma|^k$ symbols:



plus states to begin and end the process

Using finite control

- just like DFAs
- can use tuples to store different types of info
- E.g., $\{a^n b^n c^n \mid n = 1 \pmod{4} \text{ and } n = 2 \pmod{7}\}$

  States: (q, i, j) where:

  q: state from TM for $\{a^n b^n c^n \mid n \geq 1\}$

  i: counter mod 4

  j: counter mod 7

  etc.

In general, can store *any finite information* in states

Subroutine calls

- soon

"Extensions" of TMs: 2-way infinite tape

.	.	-5	-4	-3	-2	-1	0	1	2	3	4	5	.	.

Simulate with 1-way infinite tape...

0	1	-1	2	-2	3	-3	4	-4	5	-5	6	.	.	.

Must modify transitions appropriately

- remember in finite control if negative or positive
- if positive, R → RR; L → LL
- if negative, R → LL; L → RR
- must mark left edge & deal with 0 cell differently

Extension: multiple tracks

4 tracks

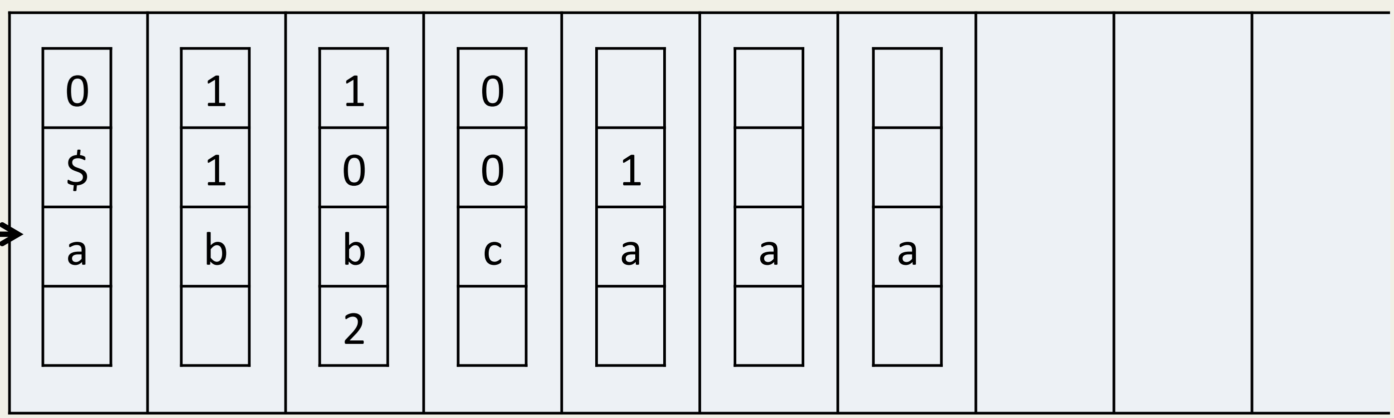
0	1	1	0						
$	1	0	0	1					
a	b	b	c	a	a	a			
		2							

infinite tape →

M can address any particular track in the cell it is scanning

Can simulate 4 tracks with a single track machine, using extra "stacked" characters:

single character



Multiple tracks

4 tracks

0	1	1	0						
$	1	0	0	1					
a	b	b	c	a	a	a			
		2							

infinite tape →

M: $\delta(q, -,0,-,-) = (p, -,-,-,1, R)$

"If in state q reading 0 on second track, then go to state p, write 1 on fourth track, and move right"

$$\text{Then in M'}\quad \delta\left(q, \begin{array}{|c|}\hline x \\ \hline 0 \\ \hline y \\ \hline z \\ \hline \end{array}\right) = \left(p, \begin{array}{|c|}\hline x \\ \hline 0 \\ \hline y \\ \hline 1 \\ \hline \end{array}, R\right)$$

for *every* x, y, z in Γ

Extension: multiple tapes

k-tape TM

- *k* different (2-way infinite) tapes
- *k* different independently controllable heads
- input initially on tape 1;   tapes 2, 3, ..., *k,* blank.
- single move:
  - read symbols under all heads
  - print (possibly different) symbols under heads
  - move all heads (possibly different directions)
  - go to new state

k-tape TM transition function

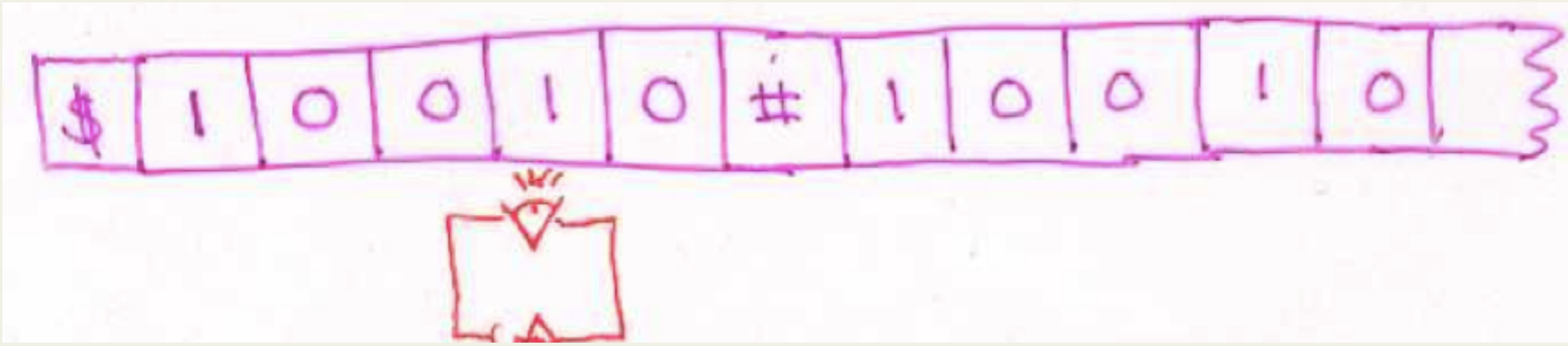
$$\delta(q, \underbrace{a_1, a_2, \dots a_k}_{\substack{\text{Symbols scanned on} \\ \text{the } k \text{ different tapes}}}) = (p, \underbrace{b_1, b_2, \dots b_k}_{\substack{\text{Symbols to be written} \\ \text{on the } k \text{ different tapes}}}, \underbrace{D_1, D_2, \dots D_k}_{\substack{\text{Directions to be moved} \\ (D_i \text{ is one of L, R, S})}})$$

Utility of multiple tapes

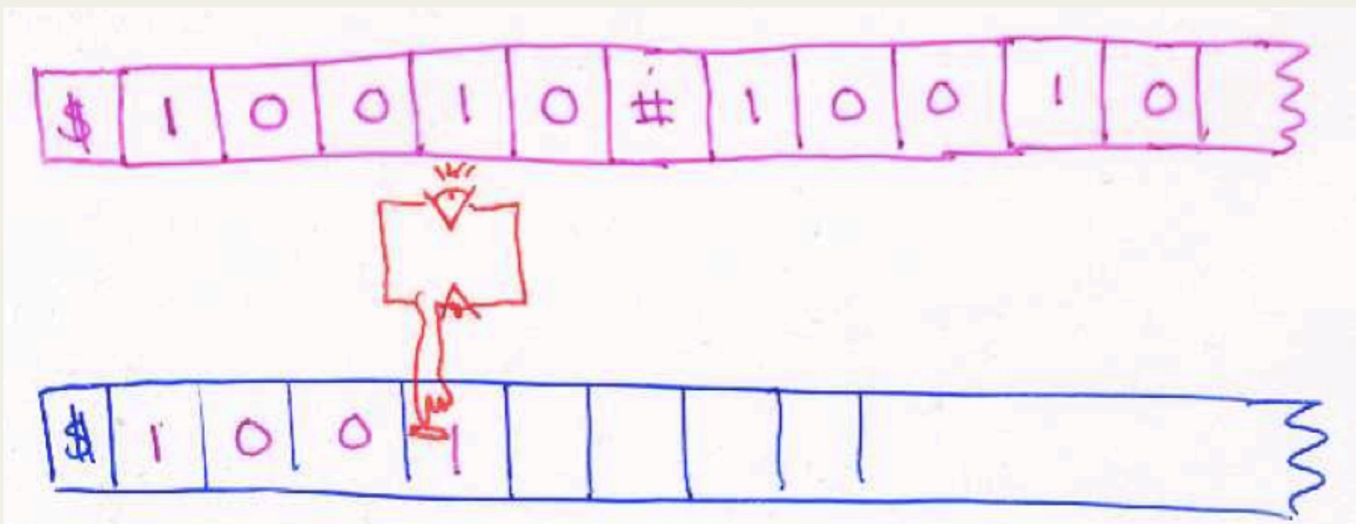
makes programming a whole lot easier

$	1	0	0	1	0	#	1	0	0	1	0			

is input string of form w#w ?



$\Omega(n^2)$ steps provably required

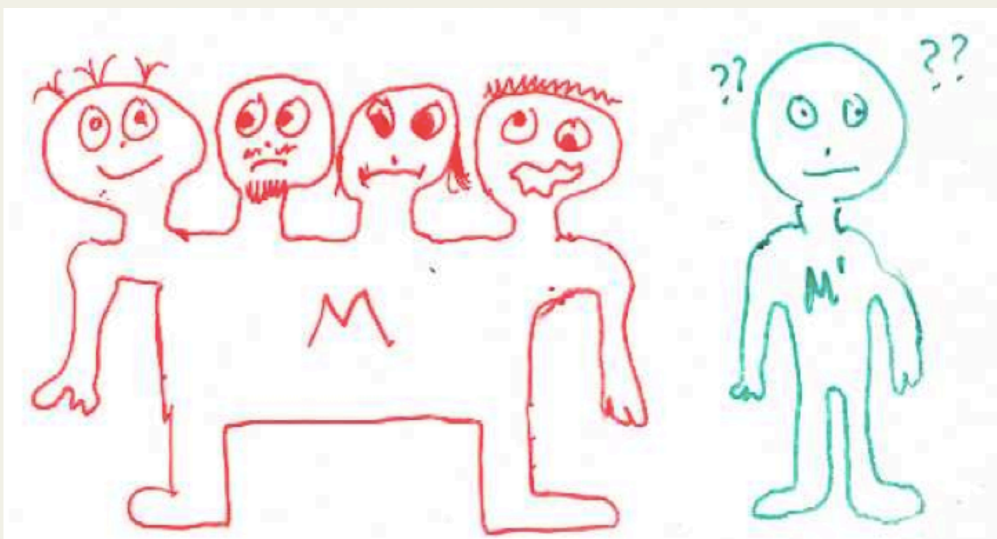


≈ 3n/2 steps easily programmed

Can't compute more with k tapes

Theorem: If L is accepted by a k-tape TM M, then L is accepted by some 1-tape TM M'.

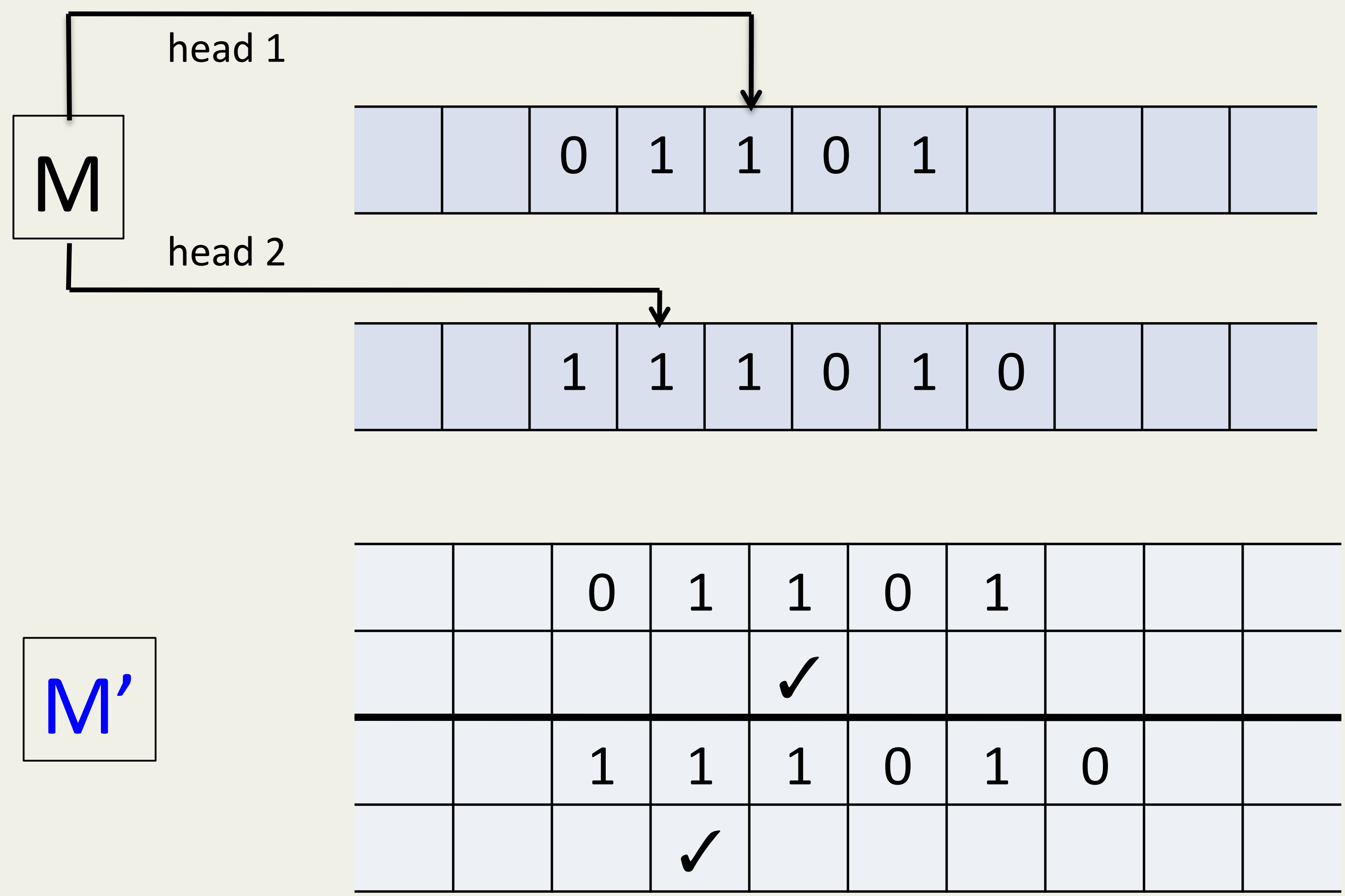
Intuition: M' uses 2k tracks to simulate M



BUT....
M has k heads!

How can M' be in k places at once?

Snapshot of simulation (k = 2)



Track 2i-1 holds tape i. Track 2i holds position of head i

To make a move, M' does:

Phase 1: Sweep from leftmost edge to rightmost "✓" on any track, noting symbols ✓'ed, and what track they are on. Save this info in finite control.

Implementation:

States of form $(q, a_1, a_2, \ldots, a_k)$, where each a_i is either "?", or the symbol scanned under the i^{th} head of *M*.

Simulation Example (k=2)

To do a "state q" move, *M'* begins in state (q, ?, ?)

("I don't know what *M* is scanning on either tape")

M' moves right over unchecked symbols until...

$$\delta'\left((q,?,?), \begin{array}{|c|}\hline 0 \\ \hline \\ \hline 1 \\ \hline \checkmark \\ \hline \end{array} \right) = \left((q,?,1), \begin{array}{|c|}\hline 0 \\ \hline \\ \hline 1 \\ \hline \checkmark \\ \hline \end{array}, R \right)$$

record info
in state

Continue moving right until state (q,a_1,a_2) is reached, with neither a_i = "?"

Simulation Example (cont.)

Phase 2: M' is in state $(q,0,1)$.

If $\delta(q,0,1) = (p,0,0,L,R)$, M' goes to state $q_{p,0,0,L,R}$ to record changes

Sweeping left, $\delta'\left(q_{p,0,0,L,R},\ \begin{bmatrix} 0 \\ \ \\ 1 \\ \checkmark \end{bmatrix}\right) = \left(q_{p,0,\checkmark,L,4},\ \begin{bmatrix} 0 \\ \ \\ 0 \\ \ \end{bmatrix},\ R\right)$

symbol has been changed

on next move, put ✓ on track 4

In next cell, $\delta'\left(q_{p,0,\checkmark,L,4},\ \begin{bmatrix} x \\ \ \\ y \\ \ \end{bmatrix}\right) = q_{p,0,\checkmark,L,\checkmark},\ \begin{bmatrix} x \\ \ \\ y \\ \checkmark \end{bmatrix},\ L)$

continue sweep

Then, M' continues left to make changes to tracks 1 & 2.

Simulation Example (cont.)

When *M'* finally reaches state $q_{p,✓,✓,✓,✓}$, it then immediately goes to state (p, ?, ?) and begins the sweep right to collect the symbols being scanned again.

Many details have been left out.

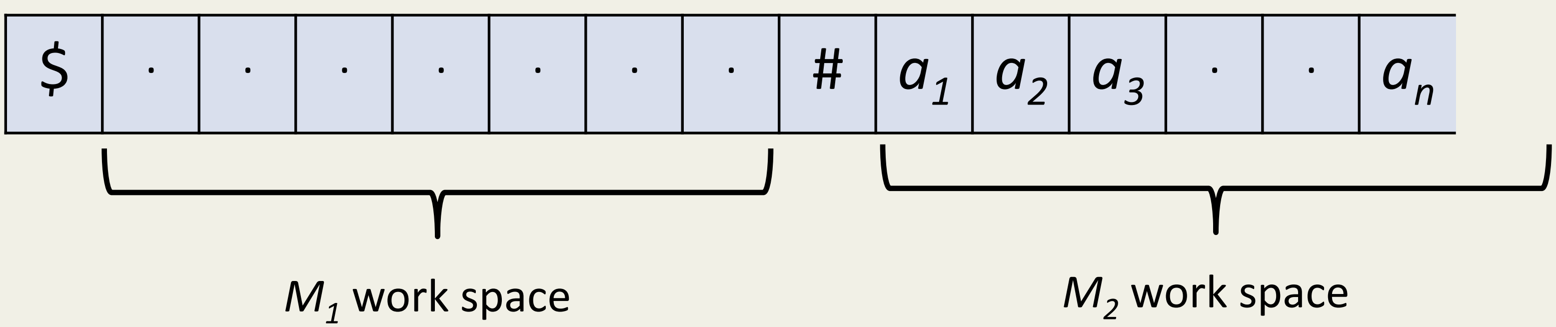
Thus, each move of *M* requires *M'* to do a complete sweep across, and back.

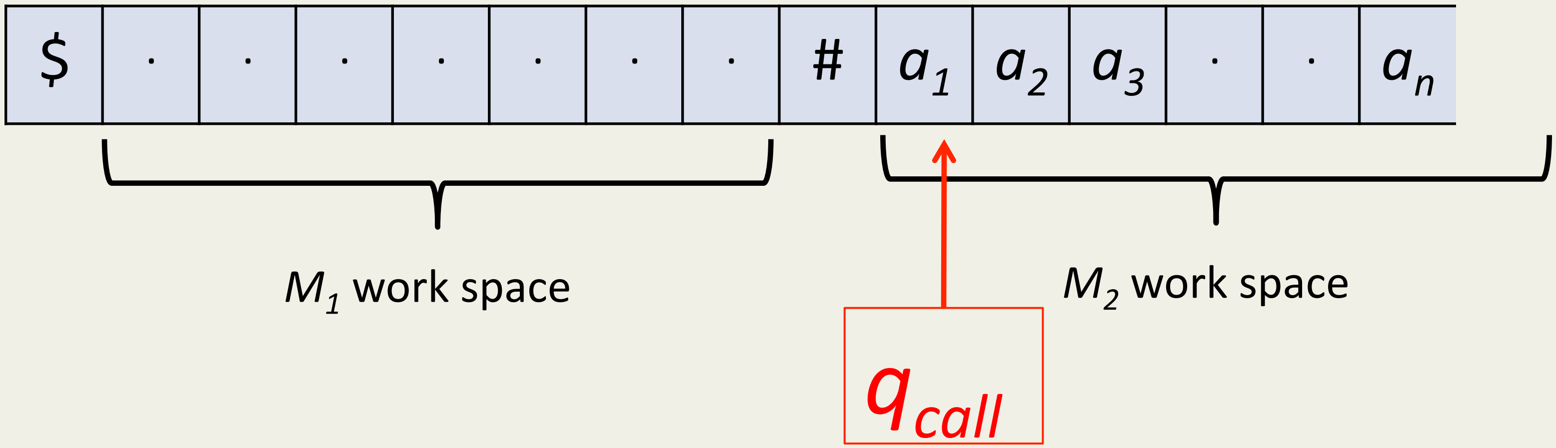
Not hard to show that if *M* takes *t* steps to complete its computation, then *M'* takes O(t^2) steps.

Subroutine calls

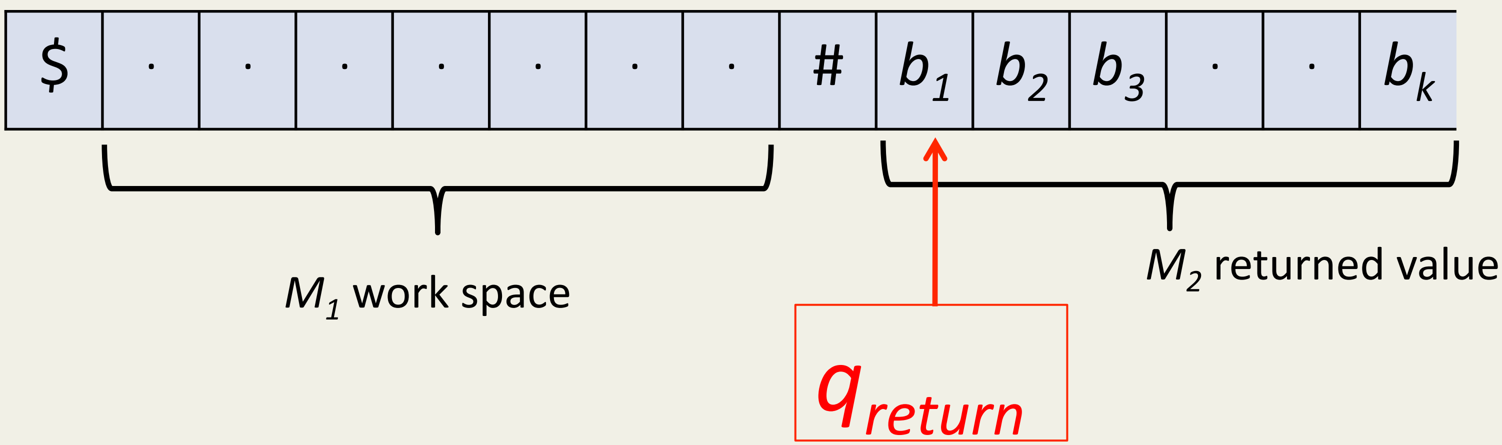
Mechanism for M_1 to "call" M_2 on an argument

- Rename states so that M_1 and M_2 have no common states *except* q_{call} and q_{return}
- Goal: M_1 calls from state q_{call} returns to q_{return}
- Rename init. state of M_2 as q_{call} ,halt state q_{return}
- M_1 sets up argument $a_1a_2...a_n$ for M_2 :

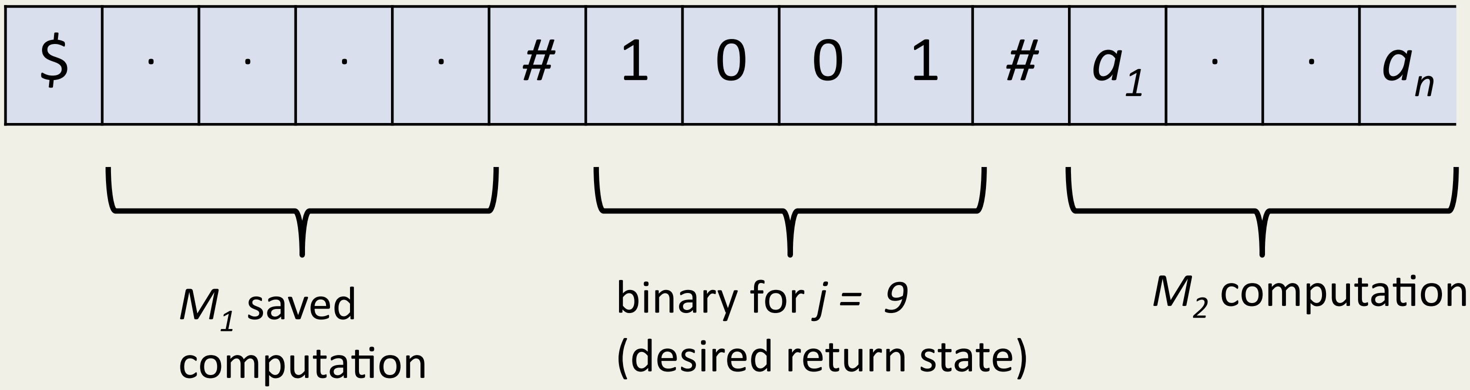




- M_2 runs, and when done:

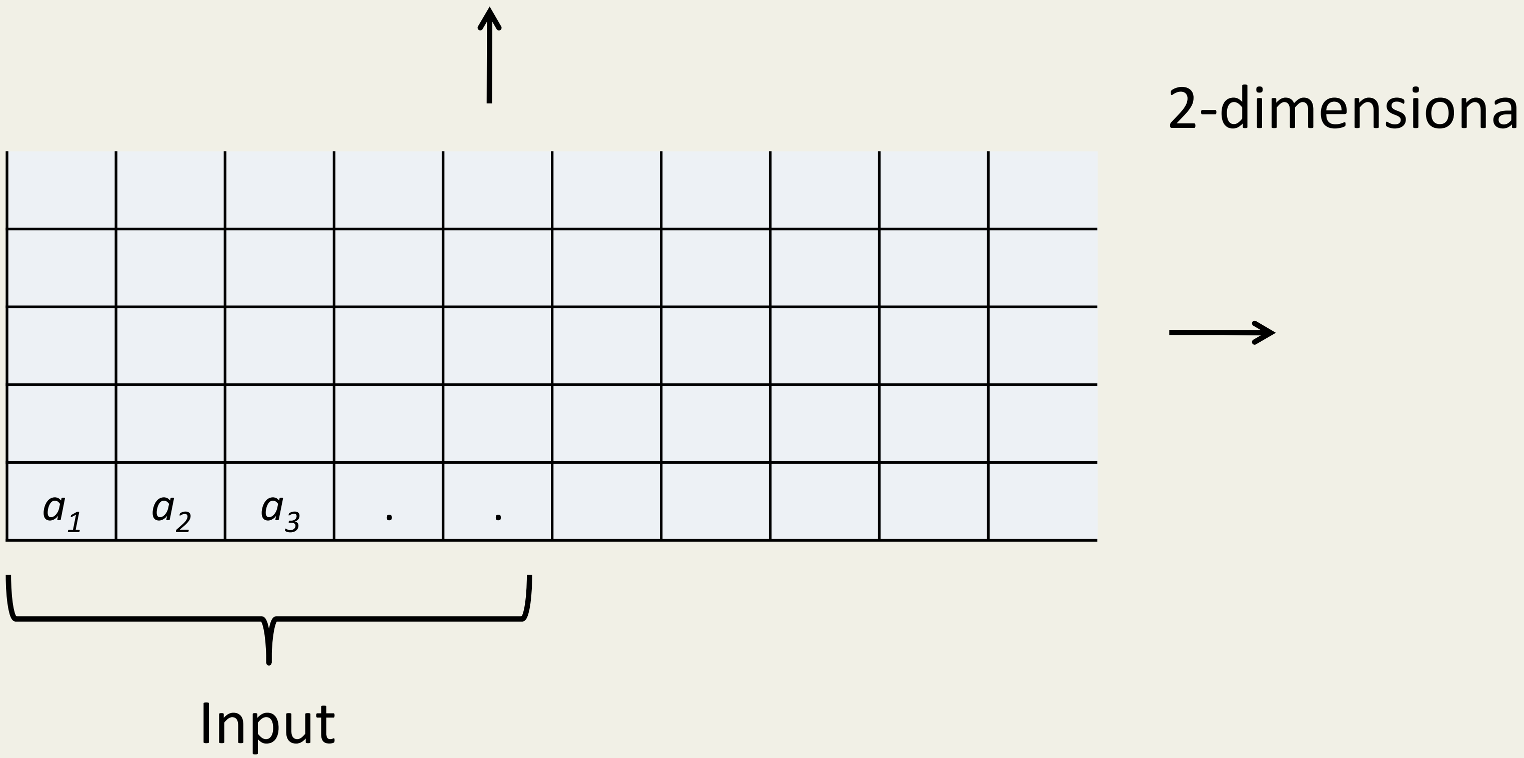


- Can be more elaborate, and return to specified state q_j



- q_{return} now goes to special sequence of states designed to read binary 1011 (or whatever) and then transition to state 9 (or whatever)
  - This can actually be done with a DFA

Multidimensional TMs



- Operates on *work grid* of *k* semi-infinite dimensions.
- Input initially along first axis.
- Transitions specify which of *2k* directions to move

Theorem: For all k, if L is accepted by a k-dim TM, then L is accepted by some 1-dim TM

Proof:

- At any time, *M* for *L* has only operated on a finite number of cells.
- In particular, in x moves, it could not have moved out of the k-dim hypercube of side x.
- *"Linearize"* the hypercube by computing offsets, exactly how arrays are stored in computer memory

Multidimensional TMs (cont)

- Example, $k = 2$

$$A[i,j] = 4(i-1) + j$$

		j		
	1	2	3	4
	5	6	7	8
i	9	10	11	12
	13	14	15	16

- If TM moves outside of current cube, remap memory to a cube with double the current sidelength, recompute layout based on new offsets