Proving Non-regularity

Lecture 6
Friday, February 7, 2020
Theorem

Languages accepted by DFAs, NFAs, and regular expressions are the same.

Question:
Is every language a regular language?
No.

Each DFA $M$ can be represented as a string over a finite alphabet by appropriate encoding. Hence the number of regular languages is countably infinite. The number of languages is uncountably infinite. Hence there must be a non-regular language!
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Regular Languages, DFAs, NFAs

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- Number of languages is *uncountably infinite*
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- Hence number of regular languages is *countably infinite*.
- Number of languages is *uncountably infinite*.
- Hence there must be a non-regular language!
Claim: Language $L$ is not regular.
How to prove non-regularity?

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**Lemma**

Consider three strings $x, y, w \in \Sigma^*$.

$M = (Q, \Sigma, \delta, s, A)$: DFA for language $L \subseteq \Sigma^*$.

If $\delta^*(s, xw) \in A$ and $\delta^*(s, yw) \notin A$ then $\delta^*(s, x) \neq \delta^*(s, y)$.
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Proof.

Assume for the sake of contradiction that $\delta^*(s, x) = \delta^*(s, y)$. 
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$\implies A \ni \delta^*(s, xw) = \delta^*(\delta^*(s, x), w)$
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$= \delta^*(s, yw) \notin A$

$\implies A \ni \delta^*(s, xw) \notin A$. Impossible!
Proof by figures

Possible

\[ \delta^*(s,x) \quad \delta^*(s,xw) \quad \delta^*(s,y) \quad \delta^*(s,yw) \]

Not possible

\[ \delta^*(s,x) = \delta^*(s,y) \quad \delta^*(s,xw) \quad \delta^*(s,yw) \]
A Simple and Canonical Non-regular Language

\[ L = \{0^i1^i \mid i \geq 0\} = \{\epsilon, 01, 0011, 000111, \cdots, \} \]

Theorem \( L \) is not regular.

Question: Proof?

Intuition: Any program to recognize \( L \) seems to require counting number of zeros in input which cannot be done with fixed memory.

How do we formalize intuition and come up with a formal proof?
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Proof by Contradiction

- Suppose $L$ is regular. Then there is a DFA $M$ such that $L(M) = L$.
- Let $M = (Q, \{0, 1\}, \delta, s, A)$ where $|Q| = n$. 

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Consider strings $\epsilon, 0, 00, 000, \cdots, 0^n$ total of $n + 1$ strings.
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Consider strings $\varepsilon, 0, 00, 000, \ldots, 0^n$ total of $n + 1$ strings.

What states does $M$ reach on the above strings? Let $q_i = \delta^*(s, 0^i)$.

By pigeon hole principle $q_i = q_j$ for some $0 \leq i < j \leq n$. That is, $M$ is in the same state after reading $0^i$ and $0^j$ where $i \neq j$. 

This contradicts the fact that $M$ accepts $L$. Thus, there is no DFA for $L$. 

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Generalizing the argument

**Definition**

For a language $L$ over $\Sigma$ and two strings $x, y \in \Sigma^*$, $x$ and $y$ are distinguishable with respect to $L$ if there is a string $w \in \Sigma^*$ such that exactly one of $xw, yw$ is in $L$.

Example:

If $i \neq j$, $0^i$ and $0^j$ are distinguishable with respect to $L = \{0^k1^k | k \geq 0\}$.

Example: $000$ and $0000$ are indistinguishable with respect to the language $L = \{w | w$ has $00$ as a substring\}.
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$x, y$ are **indistinguishable** with respect to $L$ if there is no such $w$. 

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Suppose $L = L(M)$ for some DFA $M = (Q, \Sigma, \delta, s, A)$ and suppose $x, y$ are distinguishable with respect to $L$. Then $\delta^*(s, x) \neq \delta^*(s, y)$. 
Suppose $L = L(M)$ for some DFA $M = (Q, \Sigma, \delta, s, A)$ and suppose $x, y$ are distinguishable with respect to $L$. Then $\delta^*(s, x) \neq \delta^*(s, y)$.

Proof.
Since $x, y$ are distinguishable let $w$ be the distinguishing suffix. If $\delta^*(s, x) = \delta^*(s, y)$ then $M$ will either accept both the strings $xw, yw$, or reject both. But exactly one of them is in $L$, a contradiction.
Fooling Sets

Definition

For a language $L$ over $\Sigma$ a set of strings $F$ (could be infinite) is a fooling set or distinguishing set for $L$ if every two distinct strings $x, y \in F$ are distinguishable.

Example:

$F = \{0^i | i \geq 0\}$ is a fooling set for the language $L = \{0^k 1^k | k \geq 0\}$.

Theorem

Suppose $F$ is a fooling set for $L$. If $F$ is finite then there is no DFA $M$ that accepts $L$ with less than $|F|$ states.
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Proof of Theorem

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**Proof.**

Suppose there is a DFA $M = (Q, \Sigma, \delta, s, A)$ that accepts $L$. Let $|Q| = n$. 

If $n < |F|$ then by pigeon hole principle there are two strings $x, y \in F$, $x \neq y$ such that $\delta^*(s, x) = \delta^*(s, y)$ but $x, y$ are distinguishable. Implies that there is $w$ such that exactly one of $xw, yw$ is in $L$.

However, $M$'s behavior on $xw$ and $yw$ is exactly the same and hence $M$ will accept both $xw, yw$ or reject both. A contradiction.
Suppose $F$ is a fooling set for $L$. If $F$ is finite then there is no DFA $M$ that accepts $L$ with less than $|F|$ states.

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Implies that there is $w$ such that exactly one of $xw, yw$ is in $L$. However, $M$’s behavior on $xw$ and $yw$ is exactly the same and hence $M$ will accept both $xw, yw$ or reject both. A contradiction.
**Theorem**

Suppose $F$ is a fooling set for $L$. If $F$ is finite then there is no DFA $M$ that accepts $L$ with less than $|F|$ states.

**Corollary**

If $L$ has an infinite fooling set $F$ then $L$ is not regular.
Infinite Fooling Sets

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Suppose $F$ is a fooling set for $L$. If $F$ is finite then there is no DFA $M$ that accepts $L$ with less than $|F|$ states.

Corollary
If $L$ has an infinite fooling set $F$ then $L$ is not regular.

Proof.
Suppose for contradiction that $L = L(M)$ for some DFA $M$ with $n$ states.
Any subset $F'$ of $F$ is a fooling set. (Why?) Pick $F' \subseteq F$ arbitrarily such that $|F'| > n$. By preceding theorem, we obtain a contradiction.
Examples

- \( \{0^k1^k \mid k \geq 0\} \)
Examples

- \( \{0^k1^k \mid k \geq 0\} \)
- \{bitstrings with equal number of 0s and 1s\}
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- $\{0^{k^2} \mid k \geq 0\}$
Examples

\[ \{ w w^R \mid w \in \Sigma^* \} \]
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- $\{ww^R \mid w \in \Sigma^*\}$
- $\{www \mid w \in \Sigma^*\}$
Exponential gap between NFA and DFA size

\[ L_k = \{ w \in \{0, 1\}^* \mid w \text{ has a } 1 \text{ } k \text{ positions from the end} \} \]
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Recall that \( L_k \) is accepted by a NFA \( N \) with \( k + 1 \) states.
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Recall that \( L_k \) is accepted by a NFA \( N \) with \( k + 1 \) states.

Theorem

Every DFA that accepts \( L_k \) has at least \( 2^k \) states.
Exponential gap between NFA and DFA size

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**Theorem**

Every DFA that accepts \( L_k \) has at least \( 2^k \) states.

**Claim**

\( F = \{ w \in \{0, 1\}^* : |w| = k \} \) is a fooling set of size \( 2^k \) for \( L_k \).

Why?
Exponential gap between NFA and DFA size

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Recall that $L_k$ is accepted by a NFA $N$ with $k + 1$ states.

Theorem

Every DFA that accepts $L_k$ has at least $2^k$ states.

Claim

$F = \{ w \in \{0, 1\}^* : |w| = k \}$ is a fooling set of size $2^k$ for $L_k$.

Why?

- Suppose $a_1a_2\ldots a_k$ and $b_1b_2\ldots b_k$ are two distinct bitstrings of length $k$
- Let $i$ be first index where $a_i \neq b_i$
- $y = 0^{k-i-1}$ is a distinguishing suffix for the two strings
How do we pick a fooling set $F$?

- If $x, y$ are in $F$ and $x \neq y$ they should be distinguishable! Of course.

- All strings in $F$ except maybe one should be prefixes of strings in the language $L$.

For example if $L = \{0^k1^k \mid k \geq 0\}$ do not pick 1 and 10 (say). Why?
Part I

Non-regularity via closure properties
Non-regularity via closure properties

\[ L = \{ \text{bitstrings with equal number of 0s and 1s} \} \]

\[ L' = \{ 0^k1^k \mid k \geq 0 \} \]

Suppose we have already shown that \( L' \) is non-regular. Can we show that \( L \) is non-regular without using the fooling set argument from scratch?
Non-regularity via closure properties

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Suppose we have already shown that \( L' \) is non-regular. Can we show that \( L \) is non-regular without using the fooling set argument from scratch?

\[ L' = L \cap L(0^*1^*) \]

Claim: The above and the fact that \( L' \) is non-regular implies \( L \) is non-regular. Why?
Non-regularity via closure properties

$L = \{ \text{bitstrings with equal number of 0s and 1s} \}$

$L' = \{0^k1^k \mid k \geq 0\}$

Suppose we have already shown that $L'$ is non-regular. Can we show that $L$ is non-regular without using the fooling set argument from scratch?

$L' = L \cap L(0^*1^*)$

**Claim:** The above and the fact that $L'$ is non-regular implies $L$ is non-regular. Why?

Suppose $L$ is regular. Then since $L(0^*1^*)$ is regular, and regular languages are closed under intersection, $L'$ also would be regular. But we know $L'$ is not regular, a contradiction.
Non-regularity via closure properties

General recipe:

Apply closure properties

$L_1$

$L_2$

$L_n$

$L_?$

$L_{\text{non-regular}}$
Proving non-regularity: Summary

- Method of distinguishing suffixes. To prove that $L$ is non-regular find an infinite fooling set.

- Closure properties. Use existing non-regular languages and regular languages to prove that some new language is non-regular.

- Pumping lemma. We did not cover it but it is sometimes an easier proof technique to apply, but not as general as the fooling set technique.
Part II

Myhill-Nerode Theorem
Recall:

**Definition**

For a language $L$ over $\Sigma$ and two strings $x, y \in \Sigma^*$ we say that $x$ and $y$ are **distinguishable** with respect to $L$ if there is a string $w \in \Sigma^*$ such that exactly one of $xw, yw$ is in $L$. $x, y$ are **indistinguishable** with respect to $L$ if there is no such $w$.

Given language $L$ over $\Sigma$ define a relation $\equiv_L$ over strings in $\Sigma^*$ as follows: $x \equiv_L y$ iff $x$ and $y$ are indistinguishable with respect to $L$. 
Indistinguishability

Recall:

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Given language $L$ over $\Sigma$ define a relation $\equiv_L$ over strings in $\Sigma^*$ as follows: $x \equiv_L y$ iff $x$ and $y$ are indistinguishable with respect to $L$.

**Claim**

$\equiv_L$ is an equivalence relation over $\Sigma^*$.

Therefore, $\equiv_L$ partitions $\Sigma^*$ into a collection of equivalence classes $X_1, X_2, \ldots$, 
Claim

\[ \equiv_L \text{ is an equivalence relation over } \Sigma^*. \]

Therefore, \( \equiv_L \) partitions \( \Sigma^* \) into a collection of equivalence classes.

Claim

Let \( x, y \) be two distinct strings. If \( x, y \) belong to the same equivalence class of \( \equiv_L \) then \( x, y \) are indistinguishable. Otherwise they are distinguishable.

Corollary

If \( \equiv_L \) is finite with \( n \) equivalence classes then there is a fooling set \( F \) of size \( n \) for \( L \). If \( \equiv_L \) is infinite then there is an infinite fooling set for \( L \).
Myhill-Nerode Theorem

**Theorem (Myhill-Nerode)**

$L$ is regular $\iff \equiv_L$ has a finite number of equivalence classes. If $\equiv_L$ is finite with $n$ equivalence classes then there is a DFA $M$ accepting $L$ with exactly $n$ states and this is the minimum possible.

**Corollary**

A language $L$ is non-regular if and only if there is an infinite fooling set $F$ for $L$.

**Algorithmic implication:** For every DFA $M$ one can find in polynomial time a DFA $M'$ such that $L(M) = L(M')$ and $M'$ has the fewest possible states among all such DFAs.