Non-deterministic Finite Automata (NFAs)

Lecture 4

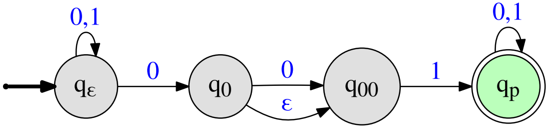
Friday, January 31, 2020

LaTeXed: January 19, 2020 04:14

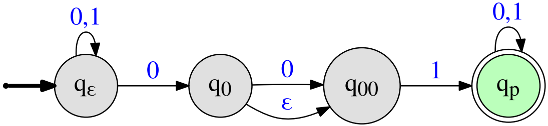
Part I

NFA Introduction

Non-deterministic Finite State Automata (NFAs)



Non-deterministic Finite State Automata (NFAs)



Differences from DFA

- From state q on same letter $a \in \Sigma$ multiple possible states
- No transitions from q on some letters
- ε-transitions!

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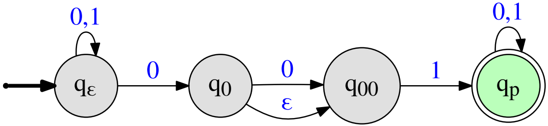


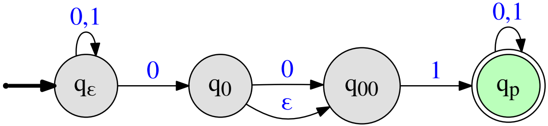
Differences from DFA

- From state q on same letter $a \in \Sigma$ multiple possible states
- No transitions from q on some letters
- ε-transitions!

Questions:

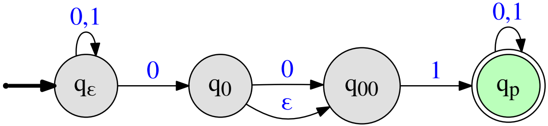
- Is this a "real" machine?
- What does it do?

NFA behavior



Machine on input string w from state q can lead to set of states (could be empty)

NFA behavior



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- From q_ε on $\mathbf{1}$

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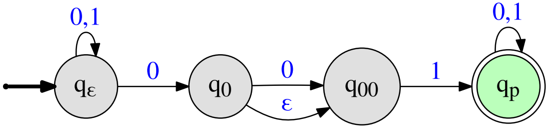
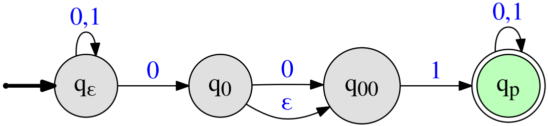


Machine on input string w from state q can lead to set of states (could be empty)

- From q_ε on $\mathbf{1}$
- From q_ε on $\mathbf{0}$

NFA behavior



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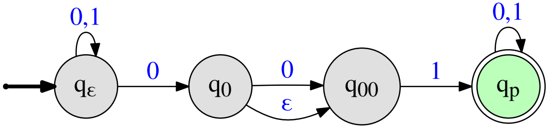
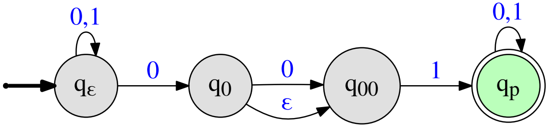


Machine on input string w from state q can lead to set of states (could be empty)

- From q_ε on **1**
- From q_ε on **0**
- From q_0 on ε
- From q_ε on **01**

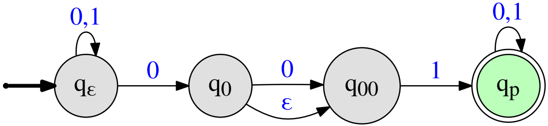
NFA behavior



Machine on input string w from state q can lead to set of states (could be empty)

- From q_ε on **1**
- From q_ε on **0**
- From q_0 on ε
- From q_ε on **01**
- From q_{00} on **00**

NFA acceptance: informal



Informal definition: An NFA N accepts a string w iff some accepting state is reached by N from the start state on input w.

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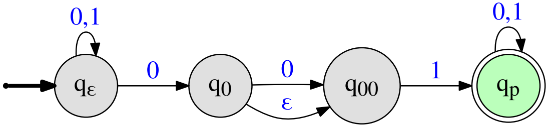
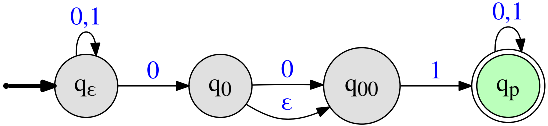


Informal definition: An NFA N accepts a string w iff some accepting state is reached by N from the start state on input w.

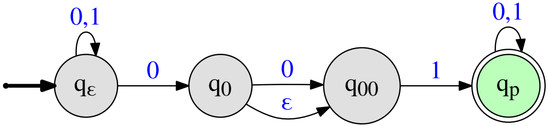
The language accepted (or recognized) by a NFA N is denote by $L(N)$ and defined as: $L(N) = \{w \mid N \text{ accepts } w\}$.

NFA acceptance: example



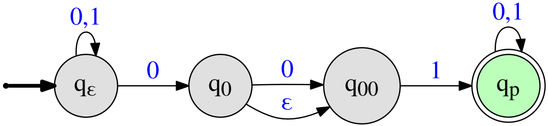
- Is **01** accepted?

NFA acceptance: example



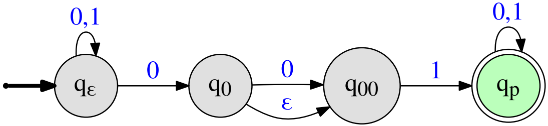
- Is **01** accepted?
- Is **001** accepted?

NFA acceptance: example



- Is **01** accepted?
- Is **001** accepted?
- Is **100** accepted?

NFA acceptance: example



- Is **01** accepted?
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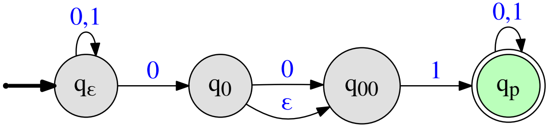
- Is **100** accepted?
- Are all strings in **1*01** accepted?

NFA acceptance: example



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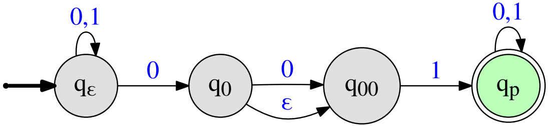
NFA acceptance: example



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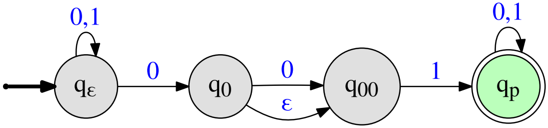


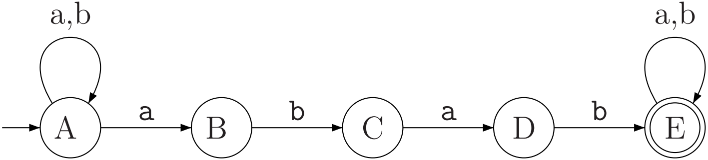
- Is **01** accepted?
- Is **001** accepted?
- Is **100** accepted?
- Are all strings in **1*01** accepted?
- What is the language accepted by N?

Comment: Unlike DFAs, it is easier in NFAs to show that a string is accepted than to show that a string is **not** accepted.

Simulating NFA

Example the first

(N1)



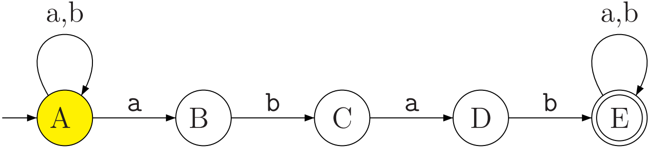
Run it on input ***ababa***.

Idea: Keep track of the states where the NFA might be at any given time.

Simulating NFA

Example the first

$t = 0$:



Remaining input: ***ababa***.

Simulating NFA

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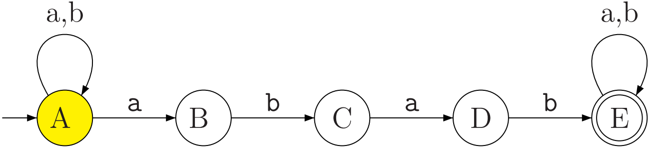
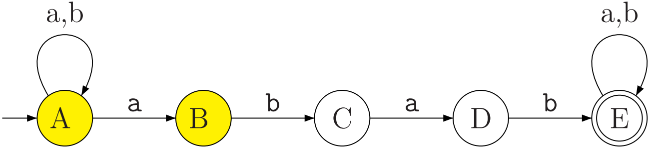
$t = 0$:



Remaining input: ***ababa***.

$t = 1$:

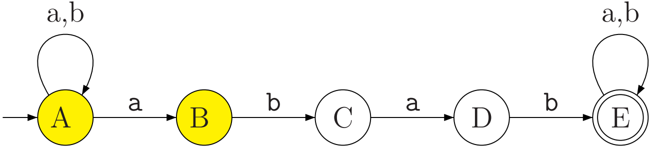


Remaining input: ***baba***.

Simulating NFA

Example the first

$t = 1$:

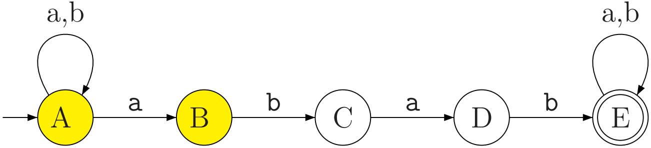


Remaining input: ***baba***.

Simulating NFA

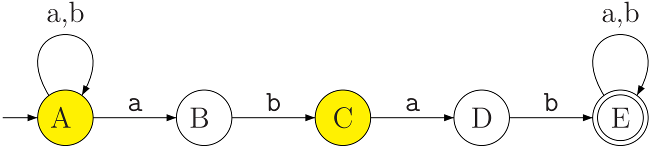
Example the first

$t = 1$:



Remaining input: ***baba***.

$t = 2$:

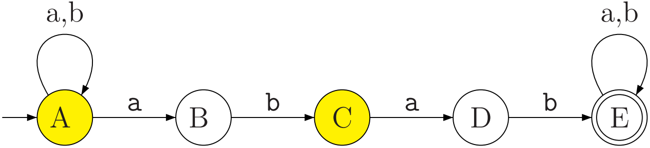


Remaining input: ***aba***.

Simulating NFA

Example the first

$t = 2$:

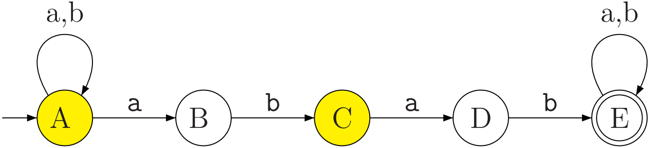


Remaining input: ***aba***.

Simulating NFA

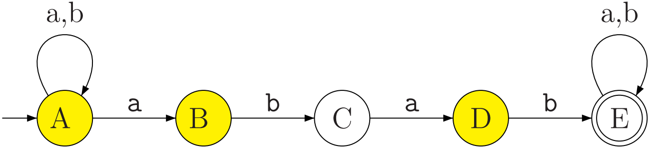
Example the first

$t = 2$:



Remaining input: ***aba***.

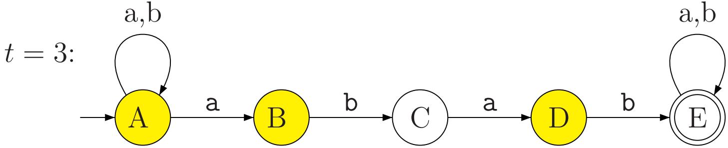
$t = 3$:



Remaining input: ***ba***.

Simulating NFA

Example the first

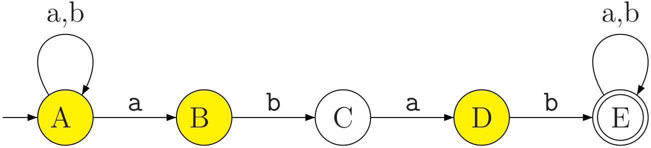


Remaining input: ***ba***.

Simulating NFA

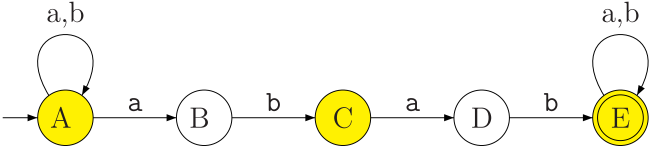
Example the first

$t = 3$:



Remaining input: ***ba***.

$t = 4$:

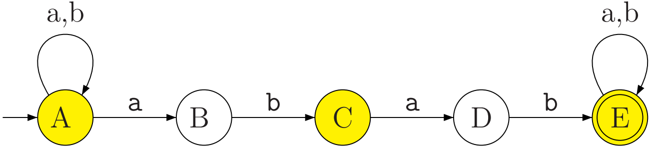


Remaining input: ***a***.

Simulating NFA

Example the first

$t = 4$:

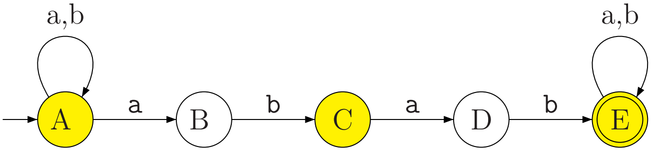


Remaining input: ***a***.

Simulating NFA

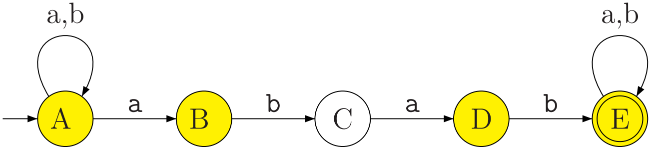
Example the first

$t = 4$:



Remaining input: ***a***.

$t = 5$:

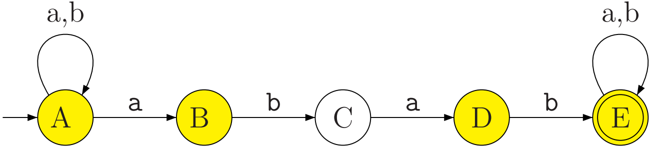


Remaining input: ε.

Simulating NFA

Example the first

$t = 5$:



Remaining input: ε.

Accepts: ***ababa***.

Formal Tuple Notation

Definition

A non-deterministic finite automata (NFA) $N = (Q, \Sigma, \delta, s, A)$ is a five tuple where

- Q is a finite set whose elements are called states,
- Σ is a finite set called the input alphabet,
- $\delta : Q \times \Sigma \cup \{\varepsilon\} \rightarrow \mathcal{P}(Q)$ is the transition function (here $\mathcal{P}(Q)$ is the power set of Q),
- $s \in Q$ is the start state,
- $A \subseteq Q$ is the set of accepting/final states.

$\delta(q, a)$ for $a \in \Sigma \cup \{\varepsilon\}$ is a subset of Q — a set of states.

Reminder: Power set

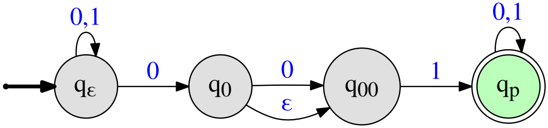
For a set Q its power set is: $\mathcal{P}(Q) = 2^Q = \{X \mid X \subseteq Q\}$ is the set of all subsets of Q.

Example

$Q = \{1, 2, 3, 4\}$

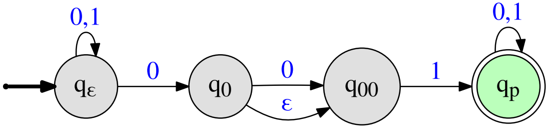
$$\mathcal{P}(Q) = \left\{ \begin{array}{c} \{1,2,3,4\}\,, \\ \{2,3,4\}\,, \{1,3,4\}\,, \{1,2,4\}\,, \{1,2,3\}\,, \\ \{1,2\}\,, \{1,3\}\,, \{1,4\}\,, \{2,3\}\,, \{2,4\}\,, \{3,4\}\,, \\ \{1\}\,, \{2\}\,, \{3\}\,, \{4\}\,, \\ \{\} \end{array} \right\}$$

Example



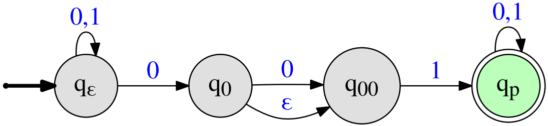
- $Q =$

Example



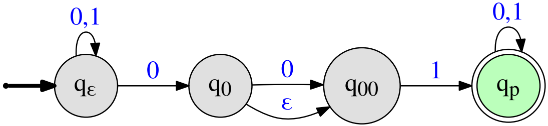
- $Q = \{q_\varepsilon, q_0, q_{00}, q_p\}$

Example



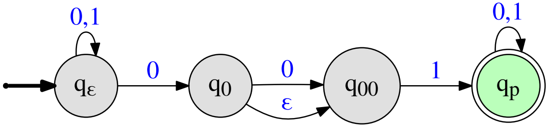
- $Q = \{q_\varepsilon, q_0, q_{00}, q_p\}$
- $\Sigma =$

Example



- $Q = \{q_\varepsilon, q_0, q_{00}, q_p\}$
- $\Sigma = \{0, 1\}$

Example



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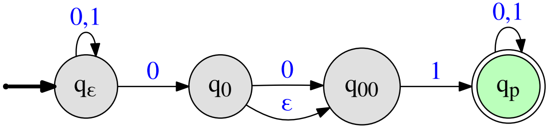
- δ

Example



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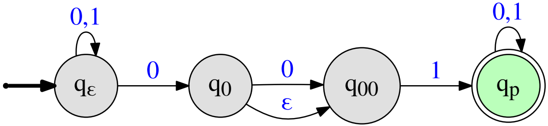

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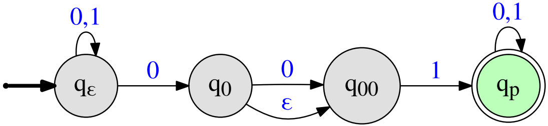
Example



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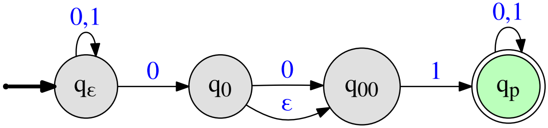

- $Q = \{q_\varepsilon, q_0, q_{00}, q_p\}$
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- $s = q_\varepsilon$
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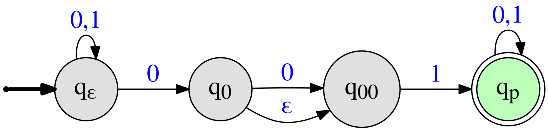
Example

Transition function in detail...



$$\delta(q_\varepsilon, \varepsilon) =$$
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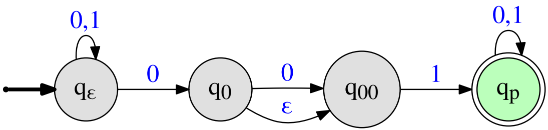

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Extending the transition function to strings

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3. Want transition function $\delta^* : Q \times \Sigma^* \rightarrow \mathcal{P}(Q)$

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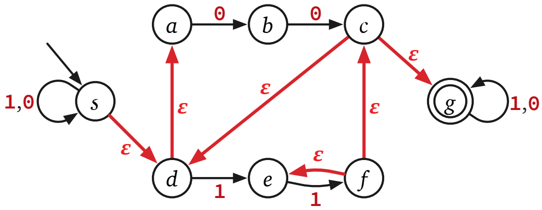
3. Want transition function $\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)$
4. $\delta^*(q, w)$: set of states reachable on input w starting in state q.

Extending the transition function to strings

Definition

For NFA $N = (Q, \Sigma, \delta, s, A)$ and $q \in Q$ the ϵreach(q) is the set of all states that q can reach using only ε-transitions.



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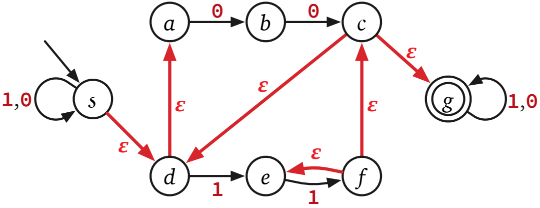

Extending the transition function to strings

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For NFA $N = (Q, \Sigma, \delta, s, A)$ and $q \in Q$ the $\epsilon\text{reach}(q)$ is the set of all states that q can reach using only ε-transitions.

Definition

Inductive definition of $\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)$:

- if $w = \varepsilon$, $\delta^*(q, w) = \epsilon\text{reach}(q)$

Extending the transition function to strings

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Definition

Inductive definition of $\delta^* : Q \times \Sigma^* \to \mathcal{P}(Q)$:

- if $w = \varepsilon$, $\delta^*(q, w) = \epsilon\text{reach}(q)$
- if $w = a$ where $a \in \Sigma$
  $\delta^*(q, a) = \cup_{p \in \epsilon\text{reach}(q)} (\cup_{r \in \delta(p,a)} \epsilon\text{reach}(r))$

Extending the transition function to strings

Definition

For NFA $N = (Q, \Sigma, \delta, s, A)$ and $q \in Q$ the $\epsilon\text{reach}(q)$ is the set of all states that q can reach using only ε-transitions.

Definition

Inductive definition of $\delta^* : Q \times \Sigma^* \rightarrow \mathcal{P}(Q)$:

- if $w = \varepsilon$, $\delta^*(q, w) = \epsilon\text{reach}(q)$
- if $w = a$ where $a \in \Sigma$
  $\delta^*(q, a) = \cup_{p \in \epsilon\text{reach}(q)} (\cup_{r \in \delta(p,a)} \epsilon\text{reach}(r))$
- if $w = ax$,
  $\delta^*(q, w) = \cup_{p \in \epsilon\text{reach}(q)} (\cup_{r \in \delta(p,a)} \delta^*(r, x))$

Formal definition of language accepted by N

Definition

A string w is accepted by NFA N if $\delta^*_N(s, w) \cap A \neq \emptyset$.

Definition

The language $L(N)$ accepted by a NFA $N = (Q, \Sigma, \delta, s, A)$ is

$$\{w \in \Sigma^* \mid \delta^*(s, w) \cap A \neq \emptyset\}.$$

Formal definition of language accepted by N

Definition

A string w is accepted by NFA N if $\delta_N^*(s, w) \cap A \neq \emptyset$.

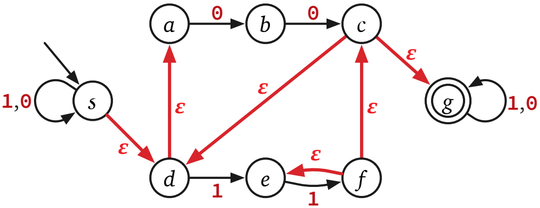
Definition

The language $L(N)$ accepted by a NFA $N = (Q, \Sigma, \delta, s, A)$ is

$$\{w \in \Sigma^* \mid \delta^*(s, w) \cap A \neq \emptyset\}.$$

Important: Formal definition of the language of NFA above uses δ^* and not δ. As such, one does not need to include ε-transitions closure when specifying δ, since δ^* takes care of that.

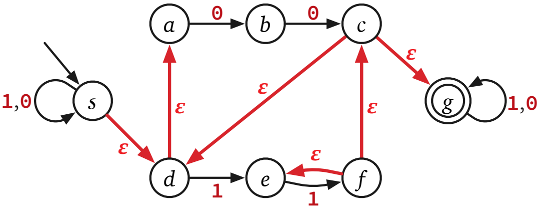
Example



What is:

- $\delta^*(s, \epsilon)$

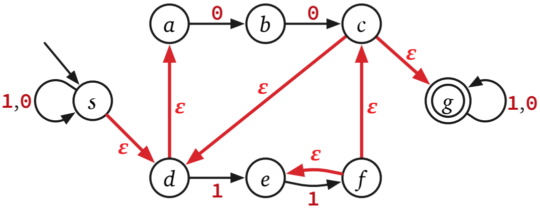
Example



What is:

- $\delta^*(s, \epsilon)$
- $\delta^*(s, 0)$

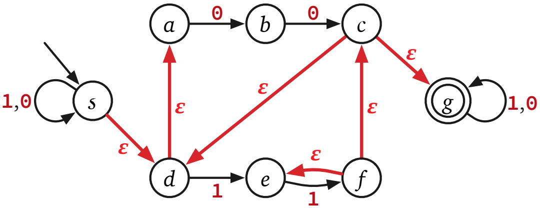
Example



What is:

- $\delta^*(s, \epsilon)$
- $\delta^*(s, 0)$
- $\delta^*(c, 0)$

Example



What is:

- $\delta^*(s, \epsilon)$
- $\delta^*(s, 0)$
- $\delta^*(c, 0)$
- $\delta^*(b, 00)$

Another definition of computation

Definition

$q \xrightarrow{w}_N p$: State p of NFA N is **reachable** from q on $w \iff$ there exists a sequence of states $r_0, r_1, \ldots, r_k$ and a sequence $x_1, x_2, \ldots, x_k$ where $x_i \in \Sigma \cup \{\varepsilon\}$, for each i, such that:

- $r_0 = q$,
- for each i, $r_{i+1} \in \delta(r_i, x_{i+1})$,
- $r_k = p$, and
- $w = x_1 x_2 x_3 \cdots x_k$.

Definition

$$\delta^* N(q, w) = \left\{ p \in Q \;\middle|\; q \xrightarrow{w}_N p \right\}.$$

Why non-determinism?

- Non-determinism adds power to the model; richer programming language and hence (much) easier to “design” programs
- Fundamental in **theory** to prove many theorems
- Very important in **practice** directly and indirectly
- Many deep connections to various fields in Computer Science and Mathematics

Many interpretations of non-determinism. Hard to understand at the outset. Get used to it and then you will appreciate it slowly.

Part II

Constructing NFAs

DFAs and NFAs

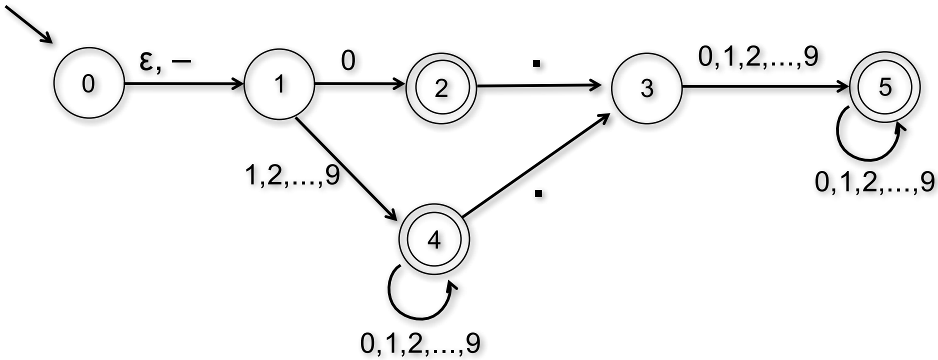
- Every DFA is a NFA so NFAs are at least as powerful as DFAs.
- NFAs prove ability to "guess and verify" which simplifies design and reduces number of states
- Easy proofs of some closure properties

Example

Strings that represent decimal numbers.

Example

Strings that represent decimal numbers.



Example

- {strings that contain CS374 as a substring}

Example

- {strings that contain CS374 as a substring}
- {strings that contain CS374 or CS473 as a substring}

Example

- {strings that contain CS374 as a substring}
- {strings that contain CS374 or CS473 as a substring}
- {strings that contain CS374 and CS473 as substrings}

Example

$L_k = \{$bitstrings that have a 1 k positions from the end$\}$

A simple transformation

Theorem

For every NFA N *there is another* NFA N' *such that* $L(N) = L(N')$ *and such that* N' *has the following two properties:*

- N' *has single final state* f *that has no outgoing transitions*
- *The start state* s *of* N *is different from* f

Part III

Closure Properties of NFAs

Closure properties of NFAs

Are the class of languages accepted by NFAs closed under the following operations?

- union
- intersection
- concatenation
- Kleene star
- complement

Closure under union

Theorem

For any two NFA*s* N_1 *and* N_2 *there is a* NFA N *such that* $L(N) = L(N_1) \cup L(N_2)$.

Closure under union

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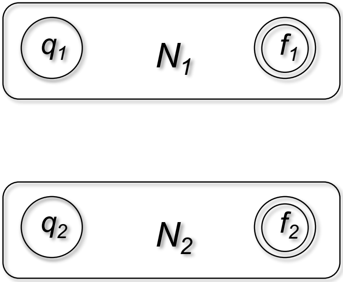

Closure under concatenation

Theorem

For any two NFAs N_1 *and* N_2 *there is a* NFA N *such that* $L(N) = L(N_1) \bullet L(N_2)$.

Closure under concatenation

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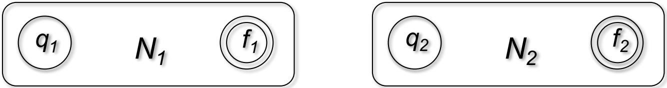
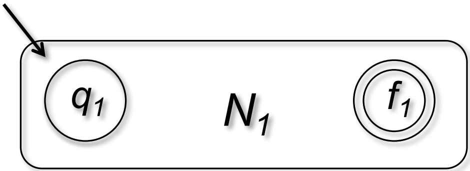


Closure under Kleene star

Theorem

For any NFA N_1 *there is a* NFA N *such that* $L(N) = (L(N_1))^*$.



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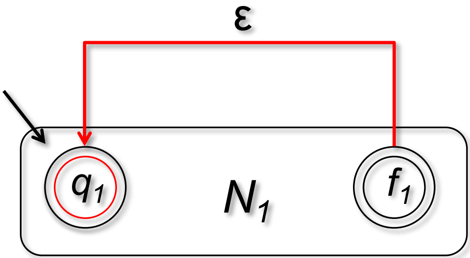

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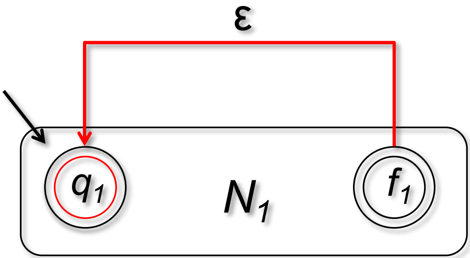

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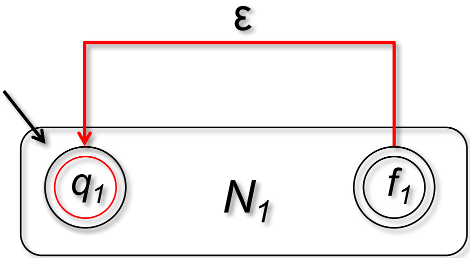

Does not work! Why?

Closure under Kleene star

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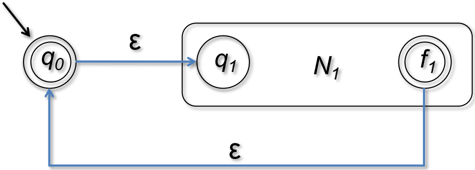

Part IV

NFAs capture Regular Languages

Regular Languages Recap

Regular Languages

$\emptyset$ regular
$\{\epsilon\}$ regular
$\{a\}$ regular for $a \in \Sigma$
$R_1 \cup R_2$ regular if both are
$R_1 R_2$ regular if both are
R^* is regular if R is

Regular Expressions

$\emptyset$ denotes $\emptyset$
ϵ denotes $\{\epsilon\}$
$\mathbf{a}$ denote $\{a\}$
$\mathbf{r}_1 + \mathbf{r}_2$ denotes $R_1 \cup R_2$
$\mathbf{r}_1 \mathbf{r}_2$ denotes $R_1 R_2$
$\mathbf{r}^*$ denote R^*

Regular expressions denote regular languages — they explicitly show the operations that were used to form the language

NFAs and Regular Language

Theorem

For every regular language L *there is an* NFA N *such that* $L = L(N)$.

NFAs and Regular Language

Theorem

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Proof strategy:

- For every regular expression r show that there is a NFA N such that $L(r) = L(N)$
- Induction on length of r

NFAs and Regular Language

- For every regular expression r show that there is a NFA N such that $L(r) = L(N)$
- Induction on length of r

Base cases: $\emptyset$, $\{\varepsilon\}$, $\{a\}$ for $a \in \Sigma$.

NFAs and Regular Language

- For every regular expression r show that there is a NFA N such that $L(r) = L(N)$
- Induction on length of r

Inductive cases:

- r_1, r_2 regular expressions and $r = r_1 + r_2$.

NFAs and Regular Language

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  By induction there are NFAs N_1, N_2 s.t
  $L(N_1) = L(r_1)$ and $L(N_2) = L(r_2)$.

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  $L(N) = L(r)$

NFAs and Regular Language

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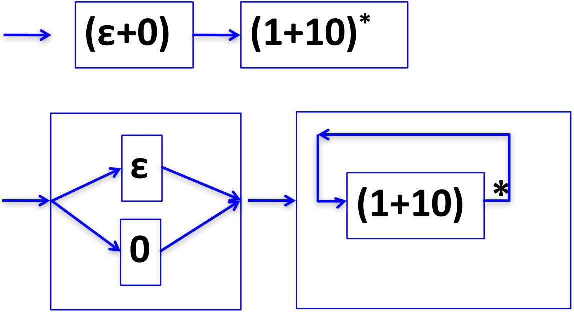
- $r = r_1 \bullet r_2$. Use closure of NFA languages under concatenation
- $r = (r_1)^*$. Use closure of NFA languages under Kleene star

Example

$(\varepsilon+0)(1+10)^*$



Example

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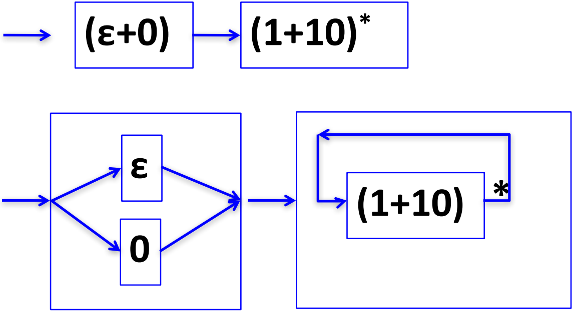
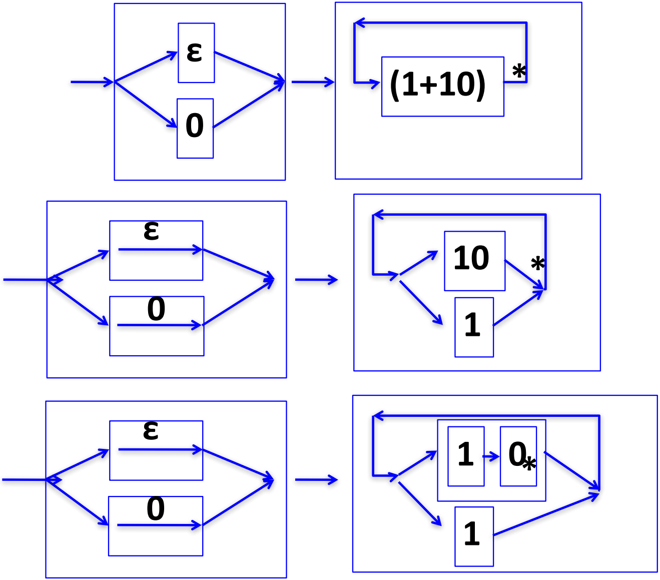


Example



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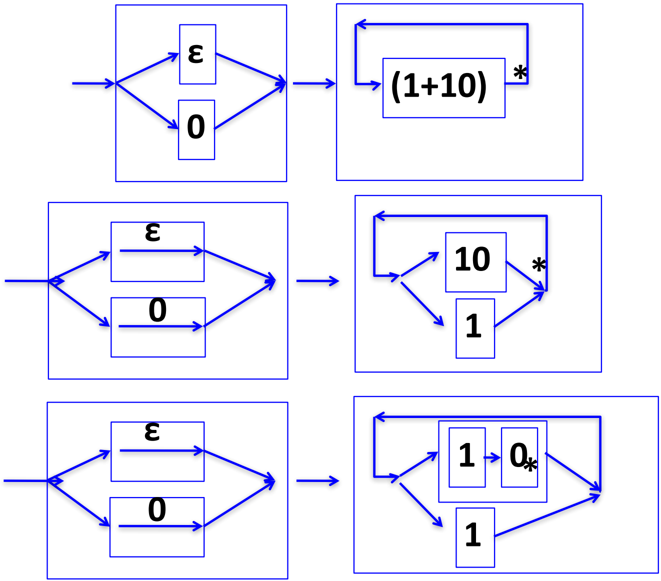

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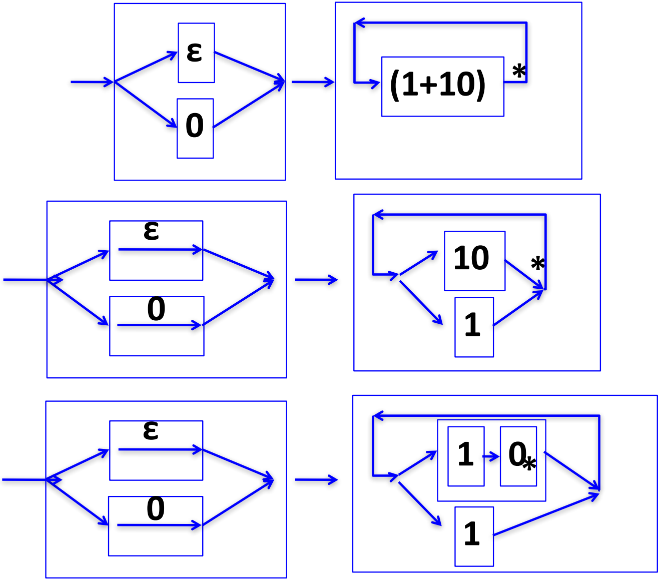

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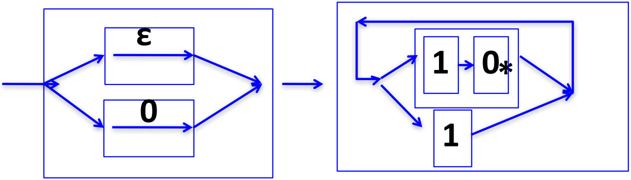

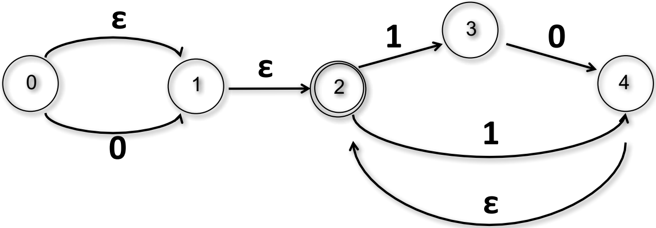

Final NFA simplified slightly to reduce states