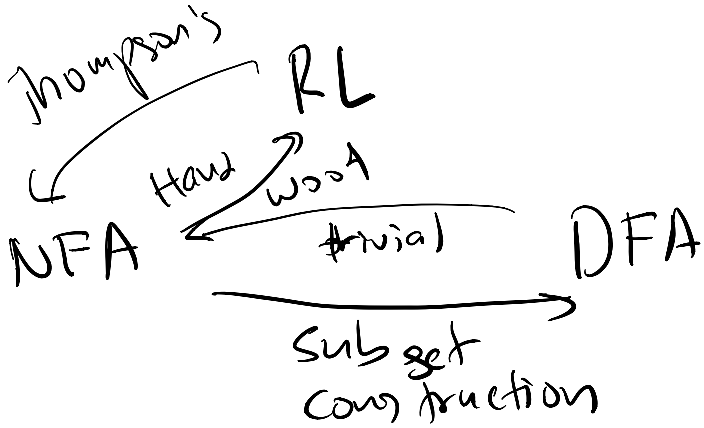
Nikita Borisov

Today

1. Non-regular languages
2. Fooling sets
3. Myhill-Nerode thm.
2.5. Closure properties



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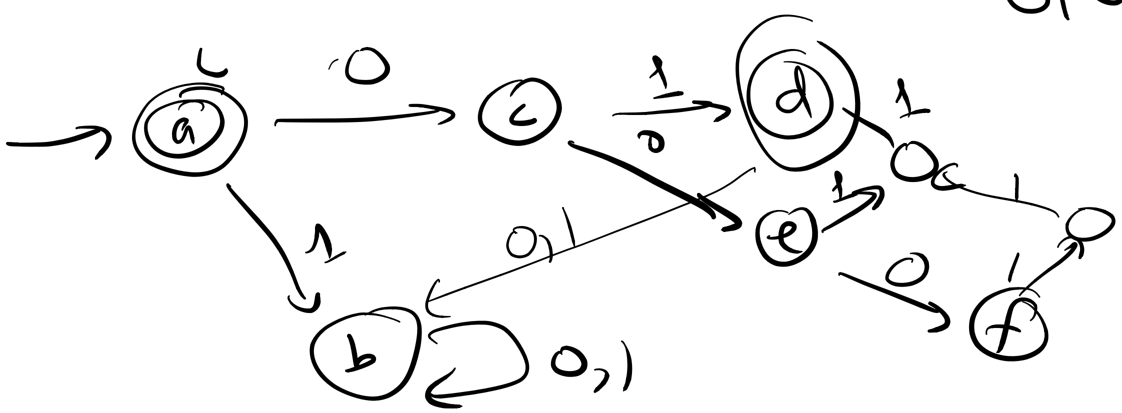
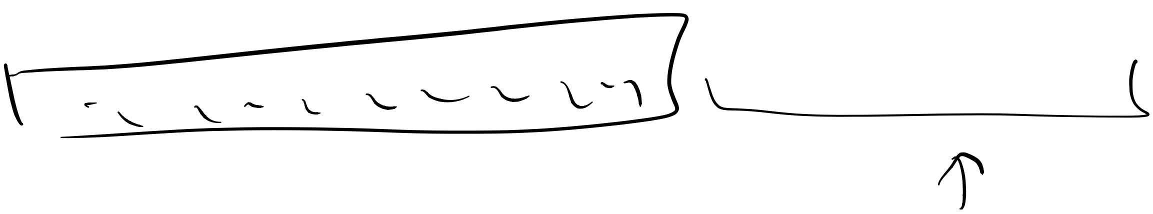
uncountably many languages
countably many RLs

Canonical non-regular language

$$\{ 0^n 1^n \mid n \geq 0 \}$$

01 ✓
0101





Lemma $M = (\Sigma, Q, \delta, s, A)$

$x, y \in \Sigma^*$, $q \in Q$

$$\delta^*(q, x \cdot y) = \delta^*(\delta^*(q, x), y)$$

Theorem $L = \{0^n 1^n\}$ is not regular

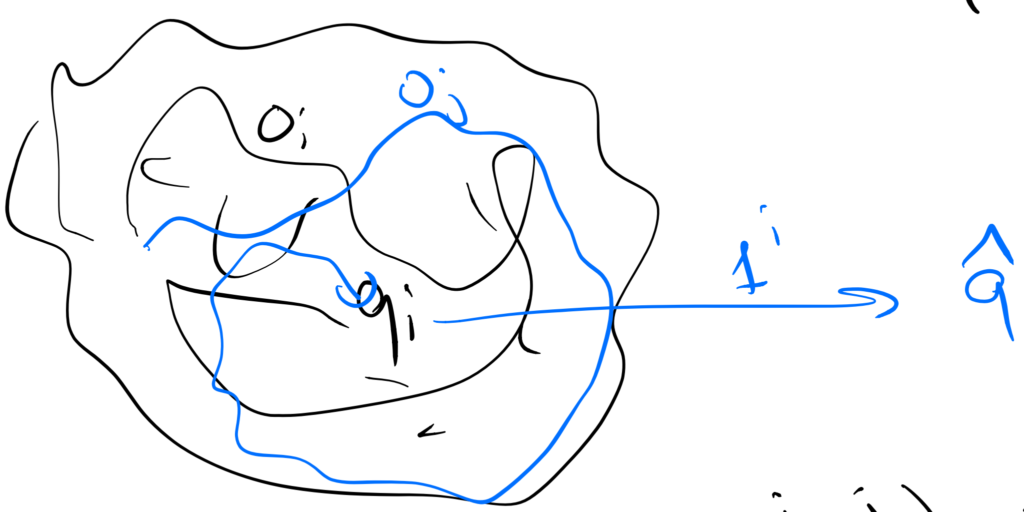
Proof Suppose o/w: $\exists\, M = (\Sigma, Q, \delta, s, A)$ with $L(M) = L$

$$\delta^*(s, 0^i) = q_i \in Q$$

Since Q is finite $\exists\, i, j$ with $i \neq j$ and $q_i = q_j$

$$\delta^*(s, 0^i 1^i) = \delta^*(\delta^*(s, 0^i), 1^i) = \delta^*(q_i, 1^i) = \hat{q} \in Q$$

$$\delta^*(s, 0^j 1^i) = \delta^*(\delta^*(s, 0^j), 1^i) = \delta^*(q_j, 1^i) = \delta^*(q_i, 1^i) = \hat{q}$$



$\hat{q} \in A$ $\quad \delta^*(s, 0^j 1^i) = \hat{q} \in A$

$\Rightarrow 0^j 1^i \in L(M)$

$i \neq j \quad 0^j 1^i \notin L$ #

$\hat{q} \notin A$ $\quad \delta^*(s, 0^i 1^i) = \hat{q} \notin A$

$\Rightarrow 0^j 1^i \notin L(M)$ #

$\therefore$ no DFA accepts $L \Rightarrow L$ is not regular

$$\begin{matrix} 0^0 & 0^1 & 0^2 & \dots & 0^i & \dots & 0^j \\ q_0 & q_1 & q_2 & & \underline{q_i} & & \underline{q_i} \end{matrix}$$

Def Given a language L, two strings x and y are distinguishable if for some suffix w $\quad xw \in L$ and $yw \notin L$ or vice versa

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For $L = \{0^n 1^n\}$ $\quad 0^i$ and 0^j are distinguishable if $i \neq j$

since 1^i is the distinguishing suffix

Def Given a language L, a set of strings F is a fooling set if for any $x, y \in F$ $\quad x, y$ are distinguishable (w r t L)

(aka distinguishing set)

$\{0^i \mid i \geq 0\}$ is a fooling set for $L = \{0^n 1^n\}$

Thm If a DFA (M) accepts language L and F is a fooling set for L then

$$\delta^*(s, x) \neq \delta^*(s, y) \text{ for } x, y \in F \quad x \neq y$$

Proof Suppose $x, y \in F$ $\quad x \neq y$

$$\delta^*(s, x) = \delta^*(s, y) = q \in Q$$

$\exists\, w \quad xw \in L \quad yw \notin L$ (wolog)

$\Rightarrow \delta^*(s, xw) = \delta^*(\delta^*(s, x), w)$

$= \delta^*(q, w)$
$= \delta^*(\delta^*(s,y), w)$
$= \delta^*(s, yw) \notin A$

Corollary 1 if we can find fooling set F for L w/ n elements, no DFA with $|Q| < n$ accepts L

Cor 2 if L has an infinite fooling set, L is not regular

Proof Suppose DFA for L w/ n states
F_{n+1} is an $n+1$-elt subset of F
$\therefore$ no DFA w/ n states accepts L #

$L = \{0^n 1^n\}$ $F = \{0^n\}$
0^i and 0^j distinguishable $i \neq j$ by 1^i

$L = \{x \in \{0,1\}^* \mid \#1(x) = \#0(x)\}$
string with same # of 1's and 0's

Try: $F = \{1^i \mid i \geq 0\}$
$x = 1^i$ $y = 1^j$ for $i \neq j$
$xw \in L$ $yw \notin L$ $w = 0^i$
$xw = 1^i 0^i$ $yw = 1^j 0^i$

L = { balanced parentheses
"()" "((()()))"
etc. }

$F = \{ (^i \mid i \geq 0\}$

$x = (^i$, $y = (^j$ $w =)^i$

$xw \in L$ $yw \notin L$

$L = \{ ww^R \mid w \in \{0,1\}^* \}$

$F =$ a aa aaa aaaa

$\underbrace{00}_{x}\ \underbrace{00}_{w}$ $\underbrace{11}_{y}\ \underbrace{00}_{w}$

$\{00^i\} \cup \{11^i\}$

$\underbrace{00}_{x}\ \underbrace{1100}_{w}$ $\underbrace{0000}_{y}\ \underbrace{1100}_{w}$

$(01)^i$ $(01)^j$

$\underbrace{(01^i)}_{x}(10)^i$ $\underbrace{(01)^j}_{y}\ \underbrace{(10)^i}_{w}$

$F = \{(01)^i \mid i \geq 0\}$

$L = \{ 0^{k^2} \mid k \geq 0 \}$

$\epsilon, 0, 0000,$
$000000000,$
$\ldots$

$xw \in L$ $yw \notin L$

0^p 0^q $p \neq q$

0^i $\qquad$ 0^j $\qquad$ $i \neq j$, $i < j$

$F \subseteq L$ $\qquad$ 0^{i^2} $\qquad$ 0^{j^2} $\qquad$ $i < j$

$\dfrac{0^{i^2}}{x}$ $\quad$ $\dfrac{0^{2i+1}}{0^{(i+1)^2}}$ $\quad$ $0^{j^2} < 0^{j^2}\, 0^{2i+1} < 0^{(j+1)^2}$

$L_k = \{ w \in \{0,1\}^* \mid w$ has a 1 as k-th last symbol $\}$

L_3 $\qquad$ $01011010\underline{1}00 \in L$

$(0+1)^*\, 1\, (0+1)^{k-1}$

$\dfrac{1}{x}\ 00$ $\qquad\qquad$ $\dfrac{0}{y}\ 00$

$\dfrac{011}{x}\ 00$ $\qquad\qquad$ $\dfrac{010}{y}\ 00$

$\dfrac{0\underline{1}1}{x}\ 0$ $\qquad\qquad$ $\dfrac{101}{y}\ 0$

$\dfrac{101}{x}\ \epsilon$ $\qquad\qquad$ $\dfrac{001}{y}\ \epsilon$

Claim $F_k = \{ w \in \{0,1\}^* \mid |w| = k \}$ is a fooling set for L_k

Proof

$x, y \in F_k \quad x \neq y$

$x = x_1 \ldots x_i \quad x_k \qquad w = 0^{i-1}$

$y = y_1 \ldots y_i \quad y_k$

$x_i = 1, \; y_i = 0$

$xw \in L_k$

$yw \notin L_k$

$\underline{0} \; \_ \; \_$

$\underline{1} \; \_ \; \_$

∎

$|F_k| = 2^k$

L_k is accepted by a $k+1$ state NFA

any DFA $\geq 2^k$ states