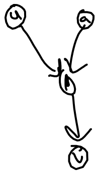
Graphs 3 (Strongly Connected Components)

Tuesday, March 31, 2020 2:51 PM

Recap:

- Undirected / Directed graphs
- Cycles / DAG



[u, a, b, v]
[a, u, b, v]

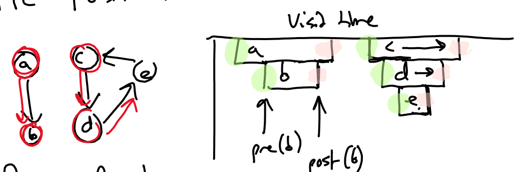
Qs about directed graphs.

- Is G a dag? - If not, then output a cycle
- Topological ordering - If so, output a topo. ordering.

- Def'n: a topological ordering $\prec$ on $G=(V,E)$ where $(u,v) \in E$ then $u \prec v$

  $u \to v$

- Pre-Post order funnel or layer diagrams.

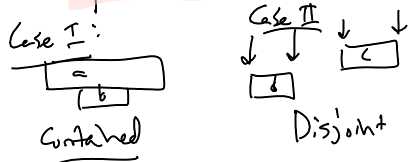



| | Pre | Post |
|---|---|---|
| a | (1, | 4) |
| b | (2, | 3) |

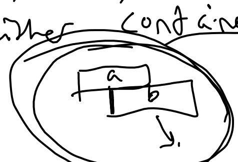
pre order: a b c d e

post order: b a e d c

Lemma:




In Any DFS, any two nodes have either contained or disjoint intervals.

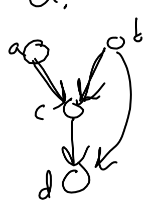


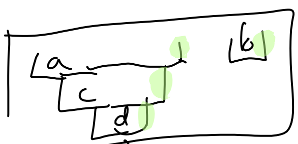
$\Rightarrow$ a ⇝ b   b ⇝ c

but a ⇝̸ c

Lemma: If G is a DAG, then rev(postorder(G)) is a topological ordering for G.

Ex.



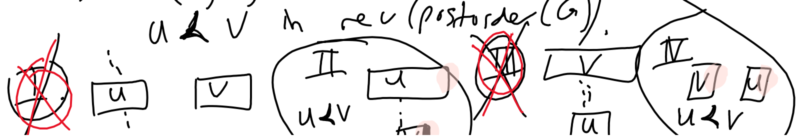


postorder(G): d, c, a, b

Claim: b a c d ✓

Proof: Let (u, v) be an edge in E. Need to show $u \prec v$ in rev(postorder(G)).

Case



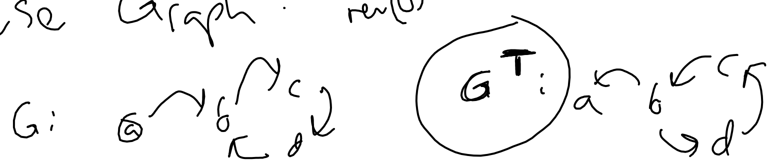
disjoint, & u before v

Case I: impossible. Because u→v.

Case III: impossible. (?) v⇝u and u→v: a cycle. but G is a dag

Both II, IV satisfy the goal. □

## Reverse Graph: rev(G) $G^{rev}$

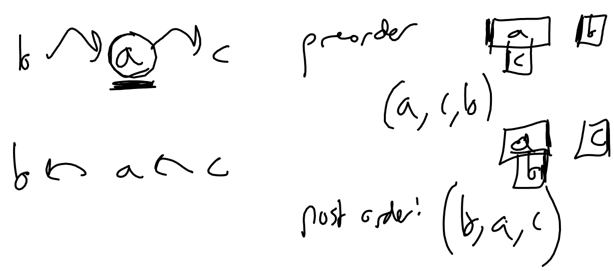



adj matrix.


$adj^T(G) = adj(G^T)$

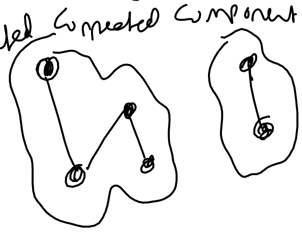
Q: - Is preorder(G) = postorder($G^T$)?

No.



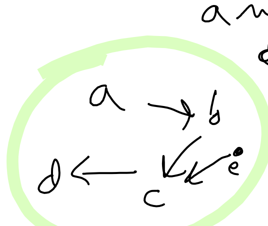

## Strongly Connected Components

Undirected Connected Components.

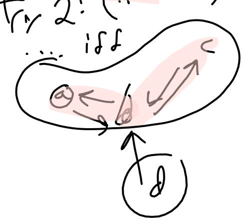


For directed graph?

Try 1: a and b are connected iff a⇝b or b⇝a




Try 2: (intersection) ... iff




Try 3: (Strongly Con. Comp)

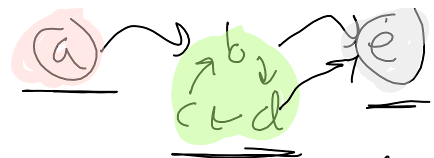
a and b are strongly connected iff a⇝b and b⇝a.


Claim: Strongly connected is an equivrelation.

Trans. $a \leadsto b$, $b \leadsto a$; $b \leadsto c$, $c \leadsto b$ $\Rightarrow$ $a \leadsto c$, $c \leadsto a$

Equiv. strongly conn. comps. are the largest sets such that every pair $u, v \in$ comp. are strongly connected.

ex.



Metagraph:

$$Meta(G) = (V', E')$$

where $V' = \{ s \in SCC(G) \}$

$E' = \{ (s,t) \in V' \mid \exists u \in s \text{ and } \exists v \in t, (u,v) \in E \}$

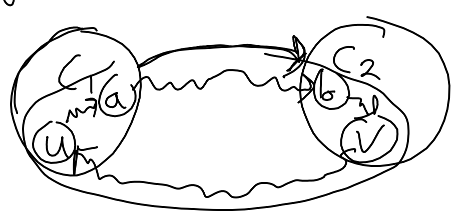
Claim: For any directed graph $G$, Meta(G) is a DAG.

Proof: Let $C_1 \rightarrow C_2$ be adjacent components in Meta(G).

Goal is to show $C_2 \leadsto C_1$

Suppose $v \in C_2$, $u \in C_1$, and $v \leadsto u$.

Then this creates a cycle. $\exists a, b \ldots$ a to v u



$u, a \in C_1$, $\Rightarrow u \leadsto a$.

$b, v \in C_2$, $b \leadsto v$

$\Rightarrow$ u and v are in the same SCC. □

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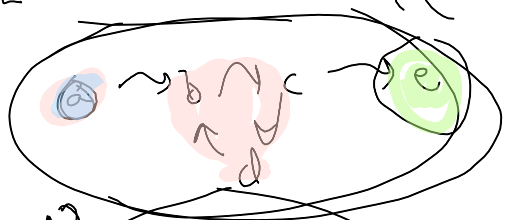
Finding Strongly Connected Components.

We can in $O(|V| + |E|) \ldots$

Easy to do in $O(|V|(|V|+|E|))$

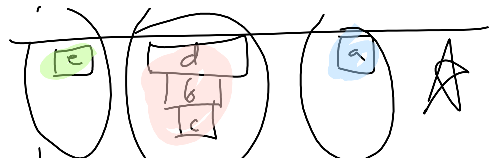



for $v \in V$:
find all nodes in reachable($v$).
for each $v \rightsquigarrow u$,
check $u \rightsquigarrow v$.

$V^2$


DFS in some order gives SCC.

In reverse order:




In alph. order


Kosaraju:

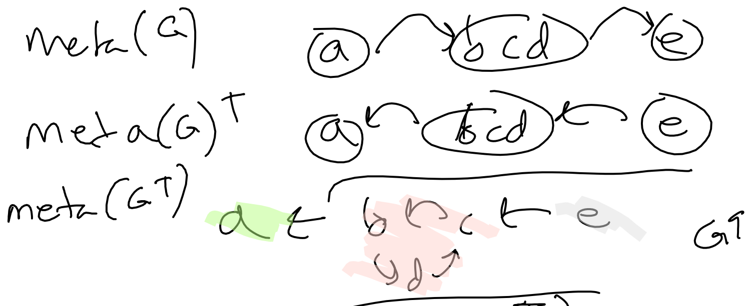
Algorithm: Two passes of DFS.

(1) Define $\prec$ by reverse postorder($G^T$)

(2) Run DFS on $G$ according to $\prec$ ordering.

$\Longrightarrow$ guaranteed to look like ★

Lemma: $\text{Meta}(G^T) = \text{Meta}(G)^T$


$SCC(G) = SCC(G^T)$

$\forall v \in C_i \in SCC(G)$

$u, v$ ...
iff $u \rightsquigarrow v$ ($u \leftrightsquigarrow v$)
$u \rightsquigarrow v \in G \Rightarrow v \rightsquigarrow u \in G^T$ ✓
$v \rightsquigarrow u \in G \Rightarrow u \rightsquigarrow v \in G^T$
$u, v \in C_1 \in SCC(G^T)$

$(C_1, C_2) \in Meta(G)$ iff
$\exists (u,v) \in G,$
$u \in C_1, v \in C_2.$
$\Rightarrow (v,u) \in G^T$
$\Rightarrow (C_2, C_1) \in Meta(G^T)$ ■

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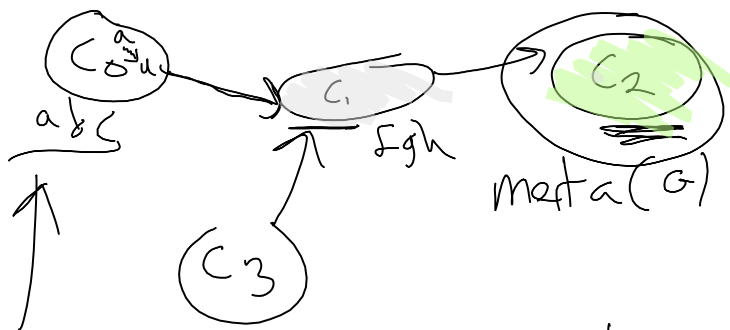
ex. 1

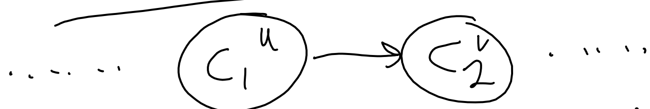



DFS explore from here.... reaches $C_2$

Lemma: let $\prec$ be ordering from Reverse Postorder($G^T$)

Let $C_1 \to C_2$ be adjacent components in meta(G).

Then $\max(C_1) \prec \max(C_2)$




$u = \max_\prec(C_1)$ $v = \max_\prec(C_2)$.

$V_\prec = [\; .... \prec u \prec .... \prec v \prec .... ]$

equiv.... $\max(C_2) > $ any node in $C_1$

Simple case: in particular, the very last node in $\prec$ is in a sink component.

$C_1 \to C_2 \in Meta(G)$
$u \in C_1$
$v = \max_\prec(C_2)$.

Proof:


4 cases for pre/post order of $u$ and $v \in G^T$

I [u] ... [v]  II [ u ] ⋮ [v]  III [ v ] ⋮ [u]  IV [v] ... [u]

Case III is impossible. $u \rightsquigarrow v \in G$, ...

Case ~~II~~ is impossible. $u \not\rightsquigarrow v \in G$ because $C_1 \neq C_2$

$\Rightarrow v \rightsquigarrow u \in G^T$ ✓

Case IV is impossible. $u \rightsquigarrow v \in G$

$\Rightarrow v \rightsquigarrow u \in G^T$

if u not visited when we explore v, we'll visit u

Case I, Case II ... satisfy $u < v$.