Prove that each of the following problems is NP-hard.

1. Given an undirected graph \( G \), does \( G \) contain a simple path that visits all but 374 vertices?

**Solution:**

We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem. Given an arbitrary graph \( G \), let \( H \) be the graph obtained from \( G \) by adding 374 isolated vertices. Call a path in \( H \) **almost-Hamiltonian** if it visits all but 374 vertices. I claim that \( G \) contains a Hamiltonian path if and only if \( H \) contains an almost-Hamiltonian path.

\[ \Rightarrow \] Suppose \( G \) has a Hamiltonian path \( P \). Then \( P \) is an almost-Hamiltonian path in \( H \), because it misses only the 374 isolated vertices.

\[ \Leftarrow \] Suppose \( H \) has an almost-Hamiltonian path \( P \). This path must miss all 374 isolated vertices in \( H \), and therefore must visit every vertex in \( G \). Every edge in \( H \), and therefore every edge in \( P \), is also an edge in \( G \). We conclude that \( P \) is a Hamiltonian path in \( G \).

Given \( G \), we can easily build \( H \) in polynomial time by brute force.

2. Given an undirected graph \( G \), does \( G \) have a spanning tree in which every node has degree at most 374?

**Solution:**

We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem. Given an arbitrary graph \( G \), let \( H \) be the graph obtained by attaching a fan of 372 edges to every vertex of \( G \). Call a spanning tree of \( H \) **almost-Hamiltonian** if it has maximum degree 374. I claim that \( G \) contains a Hamiltonian path if and only if \( H \) contains an almost-Hamiltonian spanning tree.

\[ \Rightarrow \] Suppose \( G \) has a Hamiltonian path \( P \). Let \( T \) be the spanning tree of \( H \) obtained by adding every fan edge in \( H \) to \( P \). Every vertex \( v \) of \( H \) is either a leaf of \( T \) or a vertex of \( P \). If \( v \in P \), then \( \deg_H(v) \leq 2 \), and therefore \( \deg_H(v) = \deg_P(v) + 372 \leq 374 \). We conclude that \( H \) is an almost-Hamiltonian spanning tree.

\[ \Leftarrow \] Suppose \( H \) has an almost-Hamiltonian spanning tree \( T \). The leaves of \( T \) are precisely the vertices of \( H \) with degree 1; these are also precisely the vertices of \( H \) that are not vertices of \( G \). Let \( P \) be the subtree of \( T \) obtained by deleting every leaf of \( T \). Observe that \( P \) is a spanning tree of \( G \), and for every vertex \( v \in P \), we have \( \deg_P(v) = \deg_T(v) - 372 \leq 2 \). We conclude that \( P \) is a Hamiltonian path in \( G \).

Given \( G \), we can easily build \( H \) in polynomial time by brute force.

3. Given an undirected graph \( G \), does \( G \) have a spanning tree with at most 374 leaves?

**Solution:**

We prove this problem is NP-hard by a reduction from the undirected Hamiltonian path problem. Given an arbitrary graph \( G \), let \( H \) be the graph obtained from \( G \) by adding the following vertices and edges:

- First we add a vertex \( z \) with edges to every other vertex in \( z \).
Recall that a 5-coloring of a graph $G$ is a function that assigns each vertex of $G$ a “color” from the set $\{0, 1, 2, 3, 4\}$, such that for any edge $uv$, vertices $u$ and $v$ are assigned different “colors”. A 5-coloring is careful if the colors assigned to adjacent vertices are not only distinct, but differ by more than $1 \pmod{5}$. A 5-coloring is NP-hard.

**Solution:**

We prove that careful 5-coloring is NP-hard by reduction from the standard 5COLOR problem.

Given a graph $G$, we construct a new graph $H$ by replacing each edge in $G$ with a path of length three. I claim that $H$ has a careful 5-coloring if and only if $G$ has a (not necessarily careful) 5-coloring.

$\iff$ Suppose $G$ has a 5-coloring. Consider a single edge $uv$ in $G$, and suppose $\text{color}(u) = a$ and $\text{color}(v) = b$. We color the path from $u$ to $v$ in $H$ as follows:
- If $b = (a + 1) \pmod{5}$, use colors $(a, (a + 2) \pmod{5}, (a - 1) \pmod{5}, b)$.
- If $b = (a - 1) \pmod{5}$, use colors $(a, (a - 2) \pmod{5}, (a + 1) \pmod{5}, b)$.
- Otherwise, use colors $(a, b, a, b)$.

In particular, every vertex in $G$ retains its color in $H$. The resulting 5-coloring of $H$ is careful.

$\implies$ On the other hand, suppose $H$ has a careful 5-coloring. Consider a path $(u, x, y, v)$ in $H$ corresponding to an arbitrary edge $uv$ in $G$. There are exactly eight careful colorings of this path with $\text{color}(u) = 0$, namely: $(0, 2, 0, 2), (0, 2, 0, 3), (0, 2, 4, 1), (0, 2, 4, 2), (0, 3, 0, 3), (0, 3, 0, 2), (0, 3, 1, 3), (0, 3, 1, 4)$. It follows immediately that $\text{color}(u) \neq \text{color}(v)$. Thus, if we color each vertex of $G$ with its color in $H$, we obtain a valid 5-coloring of $G$.

Given $G$, we can clearly construct $H$ in polynomial time.

Prove that the following problem is NP-hard: Given an undirected graph $G$, find any integer $k > 374$ such that $G$ has a proper coloring with $k$ colors but $G$ does not have a proper coloring with $k - 374$ colors.

**Solution:**

Let $G'$ be the union of 374 copies of $G$, with additional edges between every vertex of each copy and every vertex in every other copy. Given $G$, we can easily build $G'$ in polynomial time by brute force. Let $\chi(G)$ and $\chi(G')$ denote the minimum number of colors in any proper coloring of $G$, and define $\chi(G')$ similarly.
**6** A *bicoloring* of an undirected graph assigns each vertex a set of *two* colors. There are two types of bicoloring: In a *weak* bicoloring, the endpoints of each edge must use *different* sets of colors; however, these two sets may share one color. In a *strong* bicoloring, the endpoints of each edge must use *distinct* sets of colors; that is, they must use four colors altogether. Every strong bicoloring is also a weak bicoloring.

1. Prove that finding the minimum number of colors in a weak bicoloring of a given graph is NP-hard.

**Solution:**

It suffices to prove that deciding whether a graph has a weak bicoloring with three colors is NP-hard, using the following trivial reduction from the standard 3COLOR problem.

Let $G$ be an arbitrary undirected graph. I claim that $G$ has a proper 3-coloring if and only if $G$ has a weak bicoloring with 3 colors.

- Suppose $G$ has a proper coloring using the colors red, green, and blue. We can obtain a weak bicoloring of $G$ using only the colors cyan, magenta, and yellow by recoloring each red vertex with \{magenta, yellow\}, recoloring each blue vertex with \{magenta, cyan\}, and recoloring each green vertex with \{yellow, cyan\}.

- Suppose $G$ has a weak bicoloring using the colors cyan, magenta, yellow. Then we can obtain a proper 3-coloring of $G$ by defining red = \{magenta, yellow\}, defining blue = \{magenta, cyan\}, and defining green = \{yellow, cyan\}.

More generally, for any integer $k$ and any graph $G$, every weak $k$-bicoloring of $G$ is also a proper $\binom{k}{2}$-coloring of $G$, and vice versa.

2. Prove that finding the minimum number of colors in a strong bicoloring of a given graph is NP-hard.

**Solution:**

It suffices to prove that deciding whether a graph has a strong bicoloring with six colors is NP-hard, using the following reduction from the standard 3COLOR problem.

Let $G$ be an arbitrary undirected graph. We build a new graph $H$ from $G$ as follows:

- For every vertex $v$ in $G$, the graph $H$ contains three vertices $v_1$, $v_2$, and $v_3$ and three edges $v_1v_2$, $v_2v_3$, and $v_3v_1$.

- For every edge $uv$ in $G$, the graph $H$ contains three edges $u_1v_1$, $u_2v_2$, and $u_3v_3$.

I claim that $G$ has a proper 3-coloring if and only if $H$ has a strong bicoloring with six colors. Without loss of generality, we can assume that $G$ (and therefore $H$) is connected; otherwise, consider each component independently.

- Suppose $G$ has a proper 3-coloring with colors red, green, and blue. Then we define a strong bicoloring of $H$ with colors 1, 2, 3, 4, 5, 6 as follows:
For every red vertex $v$ in $G$, let $\text{color}(v_1) = \{1, 2\}$ and $\text{color}(v_1) = \{3, 4\}$ and $\text{color}(v_3) = \{5, 6\}$.

For every blue vertex $v$ in $G$, let $\text{color}(v_1) = \{3, 4\}$ and $\text{color}(v_1) = \{5, 6\}$ and $\text{color}(v_3) = \{1, 2\}$.

For every green vertex $v$ in $G$, let $\text{color}(v_1) = \{5, 6\}$ and $\text{color}(v_1) = \{1, 2\}$ and $\text{color}(v_3) = \{3, 4\}$.

Exhaustive case analysis confirms that every pair of adjacent vertices of $H$ has disjoint color sets.

- Suppose $H$ has a strong bicoloring with six colors. Fix an arbitrary vertex $v$ in $G$, and without loss of generality, suppose $\text{color}(v_1) = \{1, 2\}$ and $\text{color}(v_1) = \{3, 4\}$ and $\text{color}(v_3) = \{5, 6\}$.

  Exhaustive case analysis implies that for any edge $uv$, each vertex $u$ must be colored either \{1, 2\} or \{3, 4\} or \{5, 6\}. It follows by induction that every vertex in $H$ must be colored either \{1, 2\} or \{3, 4\} or \{5, 6\}.

  Now for each vertex $w$ in $G$, color $w$ red if $\text{color}(w_1) = \{1, 2\}$, blue if $\text{color}(w_1) = \{3, 4\}$, and green if $\text{color}(w_1) = \{5, 6\}$. This assignment of colors is a proper 3-coloring of $G$.

Given $G$, we can build $H$ in polynomial time by brute force.

\textit{I believe that deciding whether a graph has a strong bicoloring with five colors is also NP-hard, but I don’t have a proof yet. A graph has a strong bicoloring with four colors if and only if it is bipartite, and a strong bicoloring with two or three colors if and only if it has no edges.}