A **Hamiltonian cycle** in a graph $G$ is a cycle that goes through every vertex of $G$ exactly once. Deciding whether an arbitrary graph contains a Hamiltonian cycle is **NP-Hard**.

A **tonian cycle** in a graph $G$ is a cycle that goes through at least half of the vertices of $G$. Prove that deciding whether a graph contains a tonian cycle is **NP-Hard**.

**Solution:**

[duplicate the graph] I will describe a polynomial-time reduction from **HamiltonianCycle**. Let $G$ be an arbitrary graph (which is the given instance of **HamiltonianCycle**). Let $H$ be a graph consisting of two disjoint copies of $G$, with no edges between them; call these copies $G_1$ and $G_2$. I claim that $G$ has a Hamiltonian cycle if and only if $H$ has a tonian cycle.

$\implies$ Suppose $G$ has a Hamilton cycle $C$. Let $C_1$ be the corresponding cycle in $G_1$. $C_1$ contains exactly half of the vertices of $H$, and thus is a Hamiltonian cycle in $H$.

$\impliedby$ On the other hand, suppose $H$ has a tonian cycle $C$. Because there are no edges between the subgraphs $G_1$ and $G_2$, this cycle must lie entirely within one of these two subgraphs. $G_1$ and $G_2$ each contain exactly half the vertices of $H$, so $C$ must also contain exactly half the vertices of $H$, and thus is a Hamiltonian cycle in either $G_1$ or $G_2$. But $G_1$ and $G_2$ are just copies of $G$. We conclude that $G$ has a Hamiltonian cycle.

Given $G$, we can construct $H$ in polynomial time by brute force.

**Solution:**

[add $n$ new vertices] WE describe a polynomial-time reduction from **HamiltonianCycle**. Let $G$ be an arbitrary graph, and suppose $G$ has $n$ vertices. Let $H$ be a graph obtained by adding $n$ new vertices to $G$, but no additional edges. The claim is that $G$ has a Hamiltonian cycle $\iff H$ has a tonian cycle.

$\implies$ Suppose $G$ has a Hamiltonian cycle $C$. Then $C$ visits exactly half the vertices of $H$, and thus is a tonian cycle in $H$.

$\impliedby$ On the other hand, suppose $H$ has a tonian cycle $C$. This cycle cannot visit any of the new vertices, so it must lie entirely within the subgraph $G$. Since $G$ contains exactly half the vertices of $H$, the cycle $C$ must visit every vertex of $G$, and thus is a Hamiltonian cycle in $G$.

Given $G$, we can construct $H$ in polynomial time by brute force.

2 **Big Clique** is the following decision problem: given a graph $G = (V, E)$, does $G$ have a clique of size at least $n/2$ where $n = |V|$ is the number of nodes? Prove that **Big Clique** is NP-hard.

**Solution:**

Recall that an instance of **CLIQUE** consists of a graph $G = (V, E)$ and integer $k$. $(G, k)$ is a YES instance if $G$ has a clique of size at least $k$, otherwise it is a NO instance. For simplicity we will assume $n$ is an even number.

We describe a polynomial-time reduction from **CLIQUE** to **Big Clique**. We consider two cases depending on whether $k \leq n/2$ or not. If $k \leq n/2$ we obtain a graph $G' = (V', E')$ as follows. We add a set
of \(X\) new vertices where \(|X| = n - 2k\); thus \(V' = V \cup X\). We make \(X\) a clique by adding all possible edges between vertices of \(X\). In addition we connect each vertex \(v \in X\) to each vertex \(u \in V\). In other words \(E' = E \cup \{(u,v) \mid u \in V, v \in X\} \cup \{(a,b) \mid a,b \in X\}\). If \(k > n/2\) we let \(G' = (V', E')\) where \(V' = V \cup X\) and \(E' = E\), where \(|X| = 2k - n\). In other words we add \(2k - n\) new vertices which are isolated and have no edges incident on them.

We make the following relatively easy claims that we leave as exercises.

**Claim 0.1.** Suppose \(k \leq n/2\). Then for any clique \(S\) in \(G\), \(S \cup X\) is a clique in \(G'\). For any clique \(S' \subset G'\) the set \(S' \setminus X\) is a clique in \(G\).

**Claim 0.2.** Suppose \(k > n/2\). Then \(S\) is a clique in \(G'\) iff \(S \cap X = \emptyset\) and \(S\) is a clique in \(G\).

Now we prove the correctness of the reduction. We need to show that \(G\) has a clique of size \(k\) if and only if \(G'\) has a clique of size \(n'/2\) where \(n'\) is the number of nodes in \(G'\).

\[\iff\] Suppose \(G\) has a clique \(S\) of size \(k\). We consider two cases. If \(k > n/2\) then \(n' = n + 2k - n = 2k\); note that \(S\) is a clique in \(G'\) as well and hence \(S\) is a big clique in \(G'\) since \(|S| = k \geq n'/2\). If \(k \leq n/2\), by the first claim, \(S \cup X\) is a clique in \(G'\) of size \(k + |X| = k + n - 2k = n - k\). Moreover, \(n' = n + n - 2k = 2n - 2k\) and hence \(S \cup X\) is a big clique in \(G'\). Thus, in both cases \(G'\) has a big clique.

\[\iff\] Suppose \(G'\) has a clique of size at least \(n'/2\). Let it be \(S'\); \(|S'| \geq n'/2\). We consider two cases again. If \(k \leq n/2\), we have \(n' = 2n - 2k\) and \(|S'| \geq n - k\). By the first claim, \(S = S' \setminus X\) is a clique in \(G\). If \(|S| \geq |S'| - |X| \geq n - k - (n - 2k) \geq k\). Hence \(G\) has a clique of size \(k\). If \(k > n/2\), by the second claim \(S'\) is a clique in \(G\) and \(|S'| \geq n'/2 = (n + 2k - n)/2 = k\). Therefore, in this case as well \(G\) has a clique of size \(k\).

**Solution:**

Suppose we are given an arbitrary graph \(G\). Let \(H\) be the graph obtained from \(G\) by adding a new vertex \(a\) (called an apex) with edges to every vertex of \(G\). I claim that \(G\) is 3-colorable if and only if \(H\) is 4-colorable.

\[\implies\] Suppose \(G\) is 3-colorable. Fix an arbitrary 3-coloring of \(G\), and call the colors “red”, “green”, and “blue”. Assign the new apex \(a\) the color “plaid”. Let \(uv\) be an arbitrary edge in \(H\).

- If both \(u\) and \(v\) are vertices in \(G\), they have different colors.
- Otherwise, one endpoint of \(uv\) is plaid and the other is not, so \(u\) and \(v\) have different colors.

We conclude that we have a valid 4-coloring of \(H\), so \(H\) is 4-colorable.

\[\impliedby\] Suppose \(H\) is 4-colorable. Fix an arbitrary 4-coloring; call the apex’s color “plaid” and the other three colors “red”, “green”, and “blue”. Each edge \(uv\) in \(G\) is also an edge of \(H\) and therefore has endpoints of two different colors. Each vertex \(v\) in \(G\) is adjacent to the apex and therefore cannot be plaid. We conclude that by deleting the apex, we obtain a valid 3-coloring of \(G\), so \(G\) is 3-colorable.

We can easily transform \(G\) into \(H\) in polynomial time by brute force.

**Recall the following \(k\text{COLOR}\) problem:** Given an undirected graph \(G\), can its vertices be colored with \(k\) colors, so that every edge touches vertices with two different colors?

**3.A.** Describe a direct polynomial-time reduction from \(3\text{COLOR}\) to \(4\text{COLOR}\).
3.B. Prove that $k$COLOR problem is NP-hard for any $k \geq 3$.

Solution:

[direct] The lecture notes include a proof that 3COLOR is NP-hard. For any integer $k > 3$, I will describe a direct polynomial-time reduction from 3COLOR to $k$COLOR.

Let $G$ be an arbitrary graph. Let $H$ be the graph obtained from $G$ by adding $k - 3$ new vertices $a_1, a_2, \ldots, a_{k-3}$, each with edges to every other vertex in $H$ (including the other $a_i$’s). I claim that $G$ is 3-colorable if and only if $H$ is $k$-colorable.

$\Rightarrow$ Suppose $G$ is 3-colorable. Fix an arbitrary 3-coloring of $G$. Color the new vertices $a_1, a_2, \ldots, a_{k-3}$ with $k - 3$ new distinct colors. Every edge in $H$ is either an edge in $G$ or uses at least one new vertex $a_i$; in either case, the endpoints of the edge have different colors. We conclude that $H$ is $k$-colorable.

$\Leftarrow$ Suppose $H$ is $k$-colorable. Each vertex $a_i$ is adjacent to every other vertex in $H$, and therefore is the only vertex of its color. Thus, the vertices of $G$ use only three distinct colors. Every edge of $G$ is also an edge of $H$, so its endpoints have different colors. We conclude that the induced coloring of $G$ is a proper 3-coloring, so $G$ is 3-colorable.

Given $G$, we can construct $H$ in polynomial time by brute force.

Solution:

[induction] Let $k$ be an arbitrary integer with $k \geq 3$. Assume that $j$COLOR is NP-hard for any integer $3 \leq j < k$. There are two cases to consider.

- If $k = 3$, then $k$COLOR is NP-hard by the reduction from 3SAT in the lecture notes.
- Suppose $k = 3$. The reduction in part (a) directly generalizes to a polynomial-time reduction from $(k - 1)$COLOR to $k$COLOR: To decide whether an arbitrary graph $G$ is $(k - 1)$-colorable, add an apex and ask whether the resulting graph is $k$-colorable. The induction hypothesis implies that $(k - 1)$COLOR is NP-hard, so the reduction implies that $k$COLOR is NP-hard.

In both cases, we conclude that $k$COLOR is NP-hard.

To think about later:

4 Let $G$ be an undirected graph with weighted edges. A Hamiltonian cycle in $G$ is heavy if the total weight of edges in the cycle is at least half of the total weight of all edges in $G$. Prove that deciding whether a graph contains a heavy Hamiltonian cycle is NP-hard.

Solution:

[two new vertices] I will describe a polynomial-time a reduction from the Hamiltonian path problem. Let $G$ be an arbitrary undirected graph (without edge weights). Let $H$ be the edge-weighted graph obtained from $G$ as follows:

- Add two new vertices $s$ and $t$.
- Add edges from $s$ and $t$ to all the other vertices (including each other).
- Assign weight 1 to the edge $st$ and weight 0 to every other edge.

The total weight of all edges in $H$ is 1. Thus, a Hamiltonian cycle in $H$ is heavy if and only if it contains the edge $st$. I claim that $H$ contains a heavy Hamiltonian cycle if and only if $G$ contains a Hamiltonian path.
First, suppose $G$ has a Hamiltonian path from vertex $u$ to vertex $v$. By adding the edges $uv$, $st$, and $tu$ to this path, we obtain a Hamiltonian cycle in $H$. Moreover, this Hamiltonian cycle is heavy, because it contains the edge $st$.

On the other hand, suppose $H$ has a heavy Hamiltonian cycle. This cycle must contain the edge $st$, and therefore must visit all the other vertices in $H$ contiguously. Thus, deleting vertices $s$ and $t$ and their incident edges from the cycle leaves a Hamiltonian path in $G$.

Given $G$, we can easily construct $H$ in polynomial time by brute force.

**Solution:**

I will describe a polynomial-time a reduction from the standard Hamiltonian cycle problem. Let $G$ be an arbitrary graph (without edge weights). Let $H$ be the edge-weighted graph obtained from $G$ by assigning each edge weight 0. I claim that $H$ contains a heavy Hamiltonian cycle if and only if $G$ contains a Hamiltonian path.

Suppose $G$ has a Hamiltonian cycle $C$. The total weight of $C$ is at least half the total weight of all edges in $H$, because $0 \geq 0/2$. So $C$ is a heavy Hamiltonian cycle in $H$.

Suppose $H$ has a heavy Hamiltonian cycle $C$. By definition, $C$ is also a Hamiltonian cycle in $G$.

Given $G$, we can easily construct $H$ in polynomial time by brute force.