A subsequence of a sequence (for example, an array, linked list, or string), obtained by removing zero or more elements and keeping the rest in the same sequence order. A subsequence is called a substring if its elements are contiguous in the original sequence. For example:

- SUBSEQUENCE, SUBSEQUENCE, and the empty string $\varepsilon$ are all substrings (and therefore subsequences) of the string SUBSEQUENCE;

- SBSQNC, SQUEE, and EEE are all subsequences of SUBSEQUENCE but not substrings;

- QUEUE, EQUUS, and DIMAGGIO are not subsequences (and therefore not substrings) of SUBSEQUENCE.

Describe recursive backtracking algorithms for the following problems. Don’t worry about running times.


   For example, given the array
   $\langle 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, 2, 7 \rangle$

   your algorithm should return the integer 6, because $\langle 1, 4, 5, 6, 8, 9 \rangle$ is a longest increasing subsequence (one of many).


   For example, given the array
   $\langle 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, 2, 7 \rangle$

   your algorithm should return the integer 5, because $\langle 9, 6, 5, 4, 2 \rangle$ is a longest decreasing subsequence (one of many).


   For example, given the array
   $\langle 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, 2, 7 \rangle$

   your algorithm should return the integer 17, because $\langle 3, 1, 4, 1, 5, 2, 6, 5, 8, 7, 9, 3, 8, 4, 6, 2, 7 \rangle$ is a longest alternating subsequence (one of many).

To think about later:


   For example, given the array
   $\langle 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, 2, 7 \rangle$

   your algorithm should return the integer 6, because $\langle 3, 1, 1, 2, 5, 9 \rangle$ is a longest convex subsequence (one of many).
Given an array \( A[1..n] \), compute the length of a longest \textit{palindrome} subsequence of \( A \). Recall that a sequence \( B[1..\ell] \) is a \textit{palindrome} if \( B[i] = B[\ell - i + 1] \) for every index \( i \).

For example, given the array

\[
(3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, 8, 9, 7, 9, 3, 2, 3, 8, 4, 6, 2, 7)
\]

your algorithm should return the integer 7, because \( (4, 9, 5, 3, 5, 9, 4) \) is a longest palindrome subsequence (one of many).