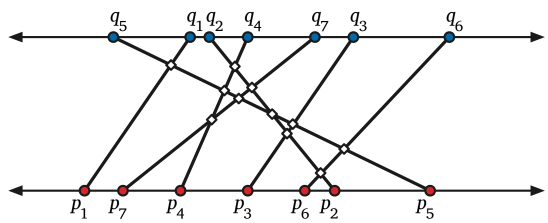
1 Suppose we are given two sets of n points, one set $\{p_1, p_2, \ldots, p_n\}$ on the line $y = 0$ and the other set $\{q_1, q_2, \ldots, q_n\}$ on the line $y = 1$. Consider the n line segments connecting each point p_i to the corresponding point q_i. Describe and analyze a divide-and-conquer algorithm to determine how many pairs of these line segments intersect, in $O(n \log n)$ time. See the example below.



Seven segments with endpoints on parallel lines, with 11 intersecting pairs.

Your input consists of two arrays $P[1..n]$ and $Q[1..n]$ of x-coordinates; you may assume that all $2n$ of these numbers are distinct. No proof of correctness is necessary, but you should justify the running time.

Solution:

We begin by sorting the array $P[1..n]$ and permuting the array $Q[1..n]$ to maintain correspondence between endpoints, in $O(n \log n)$ time. Then for any indices $i < j$, segments i and j intersect if and only if $Q[i] > Q[j]$. Thus, our goal is to compute the number of pairs of indices $i < j$ such that $Q[i] > Q[j]$. Such a pair is called an ***inversion***.

We count the number of inversions in Q using the following extension of mergesort; as a side effect, this algorithm also sorts Q. If $n < 100$, we use brute force in $O(1)$ time. Otherwise:

- Recursively count inversions in (and sort) $Q[1..\lfloor n/2 \rfloor]$.
- Recursively count inversions in (and sort) $Q[\lfloor n/2 \rfloor + 1..n]$.
- Count inversions $Q[i] > Q[j]$ where $i \le \lfloor n/2 \rfloor$ and $j > \lfloor n/2 \rfloor$ as follows:
  - Color the elements in the Left half $Q[1..n/2]$ bLue.
  - Color the elements in the Right half $Q[n/2 + 1..n]$ Red.
  - Merge $Q[1..n/2]$ and $Q[n/2 + 1..n]$, maintaining their colors.
  - For each blue element $Q[i]$, count the number of smaller red elements $Q[j]$.

The last substep can be performed in $O(n)$ time using a simple for-loop:

```
CountRedBlue(A[1..n]):
count ← 0
total ← 0
for i ← 1 to n
    if A[i] is red
        count ← count + 1
    else
        total ← total + count
return total
```

In fact, we can execute the third merge-and-count step directly by modifying the MERGE algorithm, without any need for "colors". Here changes to the standard MERGE algorithm are indicated in red.

```
MERGEANDCOUNT(A[1..n], m):
i ← 1;  j ← m + 1;  count ← 0;  total ← 0
for k ← 1 to n
   if j > n
      B[k] ← A[i];  i ← i + 1;  total ← total + count
   else if i > m
      B[k] ← A[j];  j ← j + 1;  count ← count + 1
   else if A[i] < A[j]
      B[k] ← A[i];  i ← i + 1;  total ← total + count
   else
      B[k] ← A[j];  j ← j + 1;  count ← count + 1
for k ← 1 to n
   A[k] ← B[k]
return total
```

We can further optimize this algorithm by observing that *count* is always equal to $j - m - 1$. (Proof: Initially, $j = m + 1$ and $count = 0$, and we always increment j and *count* together.)

```
MERGEANDCOUNT2(A[1..n], m):
i ← 1;  j ← m + 1;  total ← 0
for k ← 1 to n
   if j > n
      B[k] ← A[i];  i ← i + 1;  total ← total + j − m − 1
   else if i > m
      B[k] ← A[j];  j ← j + 1
   else if A[i] < A[j]
      B[k] ← A[i];  i ← i + 1;  total ← total + j − m − 1
   else
      B[k] ← A[j];  j ← j + 1
for k ← 1 to n
   A[k] ← B[k]
return total
```

The modified MERGE algorithm still runs in $O(n)$ time, so the running time of the resulting modified mergesort still obeys the recurrence $T(n) = 2T(n/2) + O(n)$. We conclude that the overall running time is $O(n \log n)$, as required.

Rubric: 10 points = 2 for base case + 3 for divide (split and recurse) + 3 for conquer (merge and count) + 2 for time analysis. Max 3 points for a correct $O(n^2)$-time algorithm. This is neither the only way to correctly describe this algorithm nor the only correct $O(n \log n)$-time algorithm. No proof of correctness is required.