HW 4: Solved Problem

Instructors: Hassanieh, Miller

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1 Let L be the set of all strings over $\{0,1\}^*$ with exactly twice as many 0s as 1s.

1.A. Describe a CFG for the language L.

(**Hint:** For any string u define $\Delta(u) = \#(0,u) - 2\#(1,u)$. Introduce intermediate variables that derive strings with $\Delta(u) = 1$ and $\Delta(u) = -1$ and use them to define a non-terminal that generates L.)

Solution:

$S \to \varepsilon \mid SS \mid 00S1 \mid 0S1S0 \mid 1S00$

1.B. Prove that your grammar G is correct. As usual, you need to prove both $L \subseteq L(G)$ and $L(G) \subseteq L$.

(**Hint:** Let $u_{\leq i}$ denote the prefix of u of length i. If $\Delta(u) = 1$, what can you say about the smallest i for which $\Delta(u_{\leq i}) = 1$? How does u split up at that position? If $\Delta(u) = -1$, what can you say about the smallest i such that $\Delta(u_{\leq i}) = -1$?)

Solution:

We separately prove $L \subseteq L(G)$ and $L(G) \subseteq L$ as follows:

Claim 4.1. *$L(G) \subseteq L$, that is, every string in $L(G)$ has exactly twice as many 0s as 1s.*

Proof: As suggested by the hint, for any string u, let $\Delta(u) = \#(0,u) - 2\#(1,u)$. We need to prove that $\Delta(w) = 0$ for every string $w \in L(G)$.

Let w be an arbitrary string in $L(G)$, and consider an arbitrary derivation of w of length k. Assume that $\Delta(x) = 0$ for every string $x \in L(G)$ that can be derived with fewer than k productions.[1] There are five cases to consider, depending on the first production in the derivation of w.

- If $w = \varepsilon$, then $\#(0,w) = \#(1,w) = 0$ by definition, so $\Delta(w) = 0$.
- Suppose the derivation begins $S \to SS \to^* w$. Then $w = xy$ for some strings $x, y \in L(G)$, each of which can be derived with fewer than k productions. The inductive hypothesis implies $\Delta(x) = \Delta(y) = 0$. It immediately follows that $\Delta(w) = 0$.[2]
- Suppose the derivation begins $S \to 00S1 \to^* w$. Then $w = 00x1$ for some string $x \in L(G)$. The inductive hypothesis implies $\Delta(x) = 0$. It immediately follows that $\Delta(w) = 0$.
- Suppose the derivation begins $S \to 1S00 \to^* w$. Then $w = 1x00$ for some string $x \in L(G)$. The inductive hypothesis implies $\Delta(x) = 0$. It immediately follows that $\Delta(w) = 0$.
- Suppose the derivation begins $S \to 0S1S1 \to^* w$. Then $w = 0x1y0$ for some strings $x, y \in L(G)$. The inductive hypothesis implies $\Delta(x) = \Delta(y) = 0$. It immediately follows that $\Delta(w) = 0$.

In all cases, we conclude that $\Delta(w) = 0$, as required. ■

Claim 4.2. *$L \subseteq L(G)$; that is, G generates every binary string with exactly twice as many 0s as 1s.*

Proof: As suggested by the hint, for any string u, let $\Delta(u) = \#(0, u) - 2\#(1, u)$. For any string u and any integer $0 \le i \le |u|$, let $\boldsymbol{u_i}$ denote the ith symbol in u, and let $\boldsymbol{u_{\le i}}$ denote the prefix of u of length i.

Let w be an arbitrary binary string with twice as many 0s as 1s. Assume that G generates every binary string x that is shorter than w and has twice as many 0s as 1s. There are two cases to consider:

- If $w = \varepsilon$, then $\varepsilon \in L(G)$ because of the production $S \to \varepsilon$.
- Suppose w is non-empty. To simplify notation, let $\Delta_i = \Delta(w_{\le i})$ for every index i, and observe that $\Delta_0 = \Delta_{|w|} = 0$. There are several subcases to consider:
  - Suppose $\Delta_i = 0$ for some index $0 < i < |w|$. Then we can write $w = xy$, where x and y are non-empty strings with $\Delta(x) = \Delta(y) = 0$. The induction hypothesis implies that $x, y \in L(G)$, and thus the production rule $S \to SS$ implies that $w \in L(G)$.
  - Suppose $\Delta_i > 0$ for all $0 < i < |w|$. Then w must begin with 00, since otherwise $\Delta_1 = -2$ or $\Delta_2 = -1$, and the last symbol in w must be 1, since otherwise $\Delta_{|w|-1} = -1$. Thus, we can write $w = 00x1$ for some binary string x. We easily observe that $\Delta(x) = 0$, so the induction hypothesis implies $x \in L(G)$, and thus the production rule $S \to 00S1$ implies $w \in L(G)$.
  - Suppose $\Delta_i < 0$ for all $0 < i < |w|$. A symmetric argument to the previous case implies $w = 1x00$ for some binary string x with $\Delta(x) = 0$. The induction hypothesis implies $x \in L(G)$, and thus the production rule $S \to 1S00$ implies $w \in L(G)$.
  - Finally, suppose none of the previous cases applies: $\Delta_i < 0$ and $\Delta_j > 0$ for some indices i and j, but $\Delta_i \ne 0$ for all $0 < i < |w|$.

    Let i be the smallest index such that $\Delta_i < 0$. Because Δ_j either increases by 1 or decreases by 2 when we increment j, for all indices $0 < j < |w|$, we must have $\Delta_j > 0$ if $j < i$ and $\Delta_j < 0$ if $j \ge i$.

    In other words, there is a *unique* index i such that $\Delta_{i-1} > 0$ and $\Delta_i < 0$. In particular, we have $\Delta_1 > 0$ and $\Delta_{|w|-1} < 0$. Thus, we can write $w = 0x1y0$ for some binary strings x and y, where $|0x1| = i$.

    We easily observe that $\Delta(x) = \Delta(y) = 0$, so the inductive hypothesis implies $x, y \in L(G)$, and thus the production rule $S \to 0S1S0$ implies $w \in L(G)$.

In all cases, we conclude that G generates w. ■

Together, Claim 1 and Claim 2 imply $L = L(G)$.

<u>*Rubric:*</u> 10 points:

- part (a) = 4 points. As usual, this is not the only correct grammar.
- part (b) = 6 points = 3 points for $\subseteq$ + 3 points for $\supseteq$, each using the standard induction template (scaled).