10. **Prove Regular**

Let $\Sigma$ be finite alphabet. A **code** is a mapping $f : \Sigma \to \{0,1\}^+$. For example, if $\Sigma = \{a,b,c\}$, a code $f$ might be $f(a) = 101$, $f(b) = 01100$, and $f(c) = 10$. (To simplify things, we assume for any $a \neq b$, we have $f(a) \neq f(b)$.)

For a string $w_1w_2 \cdots w_m \in \Sigma^*$, we define $f(w) = f(w_1)f(w_2) \cdots f(w_m)$. In the above code,

$$f(abcba) = 101 \cdot 01100 \cdot 10 \cdot 01100 \cdot 101 = 101011001001100101.$$  

10.A. (10 pts.) Let $L$ be the language of the following DFA $M$. What language does $L$ represent?

![DFA Diagram]

10.B. (20 pts.) Working on the DFA $M$ from 10.A. construct an NFA for the language $f(L)$. Here, $f(L) = \{f(w) \mid w \in L\}$ is the **code language** where $f$ is code from the above example.

10.C. (70 pts.) Let $L \subseteq \Sigma^*$ be a arbitrary regular language. Prove that the coded language $f(L) = \{f(w) \mid w \in L\}$ is regular.

Specifically, given a DFA $M = (Q, \Sigma, \delta, s, A)$ for $L$, describe how to build an NFA $N$ for $f(L)$. Then, prove the correctness of your construction, i.e., the language of the constructed NFA is indeed the desired language $f(L)$. Your construction and proof should be for any arbitrary code $f$ and not just the code in the example above. (30 points for a correct construction, and 40 points for a correct proof of correctness.)

11. **Prove Not Regular**

Prove that the following languages in 11.A. to 11.C. are not regular by providing a fooling set. You need to prove that it is an infinite fooling and valid fooling set.

11.A. (25 pts.) $L = \{0^i1^j2^k \mid i + j = k + 1\}$.

11.B. (25 pts.) $L = \{0^n3 \mid n \geq 0\}$.

11.C. (25 pts.) $L = \{0^kw\bar{w}1^k \mid 0 \leq k \leq 3, w \in \{0,1\}^+\}$, where $\bar{w}$ is the complement bit-wise not operator. For $w = w_1w_2 \cdots w_m \in \{0,1\}^*$, we define $\bar{w} = \bar{w_1}\bar{w_2} \cdots \bar{w_m}$, for $\emptyset = 1$ and $\bar{1} = 0$.

11.D. (25 pts.) Suppose $L$ is not regular. Show that $L \cup L'$ is not regular for any finite language $L'$. Give a simple example to show that $L \cup L'$ may be regular if $L'$ is infinite.
(100 pts.) Context Free Grammar

Describe a context free grammar for the following languages in 12.A. to 12.C. Clearly explain how they work and the role of each non-terminal. Unclear grammars will receive little to no credit.

12.A. (20 pts.) $L_1 = \{0^i1^j2^k3^l4^t \mid i, j, k, t \geq 0 \text{ and } i + j + k + t = \ell \}$.

12.B. (20 pts.) $L_2 = \{0, 1\}^* \setminus \{0^n1^n \mid n \geq 0 \}$, i.e., the complement of the language $\{0^n1^n \mid n \geq 0 \}$.

12.C. (20 pts.) $L_3 = \{0^i1^j2^k \mid k = 2(i + j) \}$.

12.D. (40 pts.) Prove that your grammar for $L_3$ in 12.C. is correct. You need to prove that $L_3 \subseteq L(G)$ and $L(G) \subseteq L_3$ where $G$ is your grammar from part 12.C. (See solved problem for an example of how this is done.)