1. **Review**

1.A. **(50 pts.)** Suppose $S$ is a set of 111 integers. Prove that there is a subset $S' \subseteq S$ of at least 11 numbers such that the difference of any two numbers in $S'$ is a multiple of 11.

1.B. **(50 pts.)** The famous Basque computational arborist Gorka Oihanéan has a favorite 26-node binary tree, in which each node is labeled with a letter of the alphabet. Inorder and postorder traversals of his tree visits the nodes in the following orders:


List the nodes in Professor Oihanéan’s tree according to a preorder traversal.

2. **(100 pts.) A recurrence.**

Consider the recurrence

$$T(n) = \begin{cases} T(\lfloor n/3 \rfloor) + 4T(\lfloor n/6 \rfloor) + n & n \geq 6 \\ 1 & n < 6 \end{cases}$$

Prove by induction that $T(n) = O(n \log n)$. (Recall that you need to show that $T(n) \leq c_1 n \log n + c_2$ for $n \geq 1$ where $c_1, c_2 \geq 0$ are some fixed but suitably chosen constants.)

3. **(100 pts.) Languages**

Let $L \subseteq \{0, 1\}^*$ be a language defined recursively as follows:

(i) The string 0 is in $L$.
(ii) For any string $x$ in $L$, the string $x1$ is also in $L$.
(iii) For any string $x$ in $L$, the string $1x$ is also in $L$.
(iv) For any strings $x$ and $y$ in $L$, the string $x0y$ is also in $L$.
(v) These are the only strings in $L$.

Let $\#_0(w)$ denote the number of times 0 appears in string $w$ and $\#_1(w)$ denote the number of times 1 appears in string $w$. You may assume without proof that $\#_0(xy) = \#_0(x) + \#_0(y)$, for any strings $x, y$.

3.A. **(50 pts.)** Prove by induction that every string $w \in L$ contains an odd number of 0s.

3.B. **(50 pts.)** Let $L' \subseteq \{0, 1\}^*$ be the language of strings with an odd number of 0s. Prove that $L = L'$.