Prove that each of the following languages is \textit{not} regular.

1. \{0^{2n} \mid n \geq 0\}

Solution:
Let \(F = L = \{0^{2n} \mid n \geq 0\}\).
Let \(x\) and \(y\) be arbitrary elements of \(F\).
Then \(x = 0^{2^i}\) and \(y = 0^{2^j}\) for some non-negative integers \(x\) and \(y\).
Let \(z = 0^{2^i}\).
Then \(xz = 0^{2^i}0^{2^i} = 0^{2^{i+1}} \in L\).
And \(yz = 0^{2^j}0^{2^i} = 0^{2^{i+2^j}} \notin L\), because \(i \neq j\).
Thus, \(F\) is a fooling set for \(L\).
Because \(F\) is infinite, \(L\) cannot be regular.

2. \{0^{2n}1^n \mid n \geq 0\}

Solution:
For any non-negative integers \(i \neq j\), the strings \(0^{2^i}\) and \(0^{2^j}\) are distinguished by the suffix \(0^{2^i}\), because \(0^{2^i}0^{2^j} = 0^{2^{i+1}} \in L\) but \(0^{2^i}0^{2^j} = 0^{2^{i+2^j}} \notin L\). Thus \(L\) itself is an infinite fooling set for \(L\).

Solution:
Let \(F\) be the language \(0^*\).
Let \(x\) and \(y\) be arbitrary strings in \(F\).
Then \(x = 0^i\) and \(y = 0^j\) for some non-negative integers \(i \neq j\).
Let \(z = 0^{i}1^i\).
Then \(xz = 0^{2i}1^i \in L\).
And \(yz = 0^{i+j}1^i \notin L\), because \(i + j \neq 2i\).
Thus, \(F\) is a fooling set for \(L\).
Because \(F\) is infinite, \(L\) cannot be regular.

Solution:
For all non-negative integers \(i \neq j\), the strings \(0^i\) and \(0^j\) are distinguished by the suffix \(0^i1^i\), because \(0^{2i}1^i \in L\) but \(0^{i+j}1^i \notin L\). Thus, the language \(0^*\) is an infinite fooling set for \(L\).
Solution:
For all non-negative integers \( i \neq j \), the strings \( 0^{2i} \) and \( 0^{2j} \) are distinguished by the suffix \( 1^i \), because \( 0^{2i}1^i \in L \) but \( 0^{2j}1^i \notin L \). Thus, the language \( (00)^* \) is an infinite fooling set for \( L \).

3 \{0^m1^n \mid m \neq 2n\}

Solution:
Let \( F \) be the language \( 0^* \).
Let \( x \) and \( y \) be arbitrary strings in \( F \).
Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).
Let \( z = 0^i1^i \).
Then \( xz = 0^{2i}1^i \notin L \).
And \( yz = 0^{i+j}1^i \in L \), because \( i + j \neq 2i \).
Thus, \( F \) is a fooling set for \( L \).
Because \( F \) is infinite, \( L \) cannot be regular.

Solution:
For all non-negative integers \( i \neq j \), the strings \( 0^{2i} \) and \( 0^{2j} \) are distinguished by the suffix \( 1^i \), because \( 0^{2i}1^i \notin L \) but \( 0^{2j}1^i \in L \). Thus, the language \( (00)^* \) is an infinite fooling set for \( L \).

4 Strings over \{0,1\} where the number of 0s is exactly twice the number of 1s.

Solution:
Let \( F \) be the language \( 0^* \).
Let \( x \) and \( y \) be arbitrary strings in \( F \).
Then \( x = 0^i \) and \( y = 0^j \) for some non-negative integers \( i \neq j \).
Let \( z = 0^i1^i \).
Then \( xz = 0^{2i}1^i \in L \).
And \( yz = 0^{i+j}1^i \notin L \), because \( i + j \neq 2i \).
Thus, \( F \) is a fooling set for \( L \).
Because \( F \) is infinite, \( L \) cannot be regular.

Solution:
For all non-negative integers \( i \neq j \), the strings \( 0^{2i} \) and \( 0^{2j} \) are distinguished by the suffix \( 1^i \), because \( 0^{2i}1^i \in L \) but \( 0^{2j}1^i \notin L \). Thus, the language \( (00)^* \) is an infinite fooling set for \( L \).
Solution:
If $L$ were regular, then the language

$$(0 + 1)^* \setminus L \cap 0^*1^* = \{0^n1^n \mid m \neq 2n\}$$

would also be regular, because regular languages are closed under complement and intersection. But we just proved that $\{0^n1^n \mid m \neq 2n\}$ is not regular in problem 3. [Yes, this proof would be worth full credit, either in homework or on an exam.]

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5 Strings of properly nested parentheses (), brackets [], and braces {}. For example, the string ([]){} is in this language, but the string ( [ ) ] is not, because the left and right delimiters don’t match.

Solution:
Let $F$ be the language $0^*$. Let $x$ and $y$ be arbitrary strings in $F$.
Then $x = (i)$ and $y = (j)$ for some non-negative integers $i \neq j$.
Let $z = #i$.
Then $xz = (i)^i \in L$.
And $yz = (i)^j \notin L$, because $i \neq j$.
Thus, $F$ is a fooling set for $L$.
Because $F$ is infinite, $L$ cannot be regular.

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Solution:
For any non-negative integers $i \neq j$, the strings $(i)$ and $(j)$ are distinguished by the suffix $(i)$, because $(i)^i \in L$ but $(i)^j \notin L$. Thus, the language $(*)$ is an infinite fooling set.

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6 Strings of the form $w_1\#w_2\#\cdots\#w_n$ for some $n \geq 2$, where each substring $w_i$ is a string in $\{0,1\}^*$, and some pair of substrings $w_i$ and $w_j$ are equal.

Solution:
Let $F$ be the language $0^*$.
Let $x$ and $y$ be arbitrary strings in $F$.
Then $x = 0^i$ and $y = 0^j$ for some non-negative integers $i \neq j$.
Let $z = \#0^i$.
Then $xz = 0^i\#0^i \in L$.
And $yz = 0^j\#0^i \notin L$, because $i \neq j$.
Thus, $F$ is a fooling set for $L$.
Because $F$ is infinite, $L$ cannot be regular.
Solution:
For any non-negative integers $i \neq j$, the strings $0^i$ and $0^j$ are distinguished by the suffix $\#0^i$, because $0^i \#0^j \in L$ but $0^j \#0^i \notin L$. Thus, the language $0^* \#0^* \notin L$. Thus, the language $\#0^*$ is an infinite fooling set.

Extra problems

7. \{0^{n^2} \mid n \geq 0\}

Solution:
Let $x$ and $y$ be distinct arbitrary strings in $L$. Without loss of generality, $x = 0^{i^2}$ and $y = 0^{j^2}$ for some $i > j \geq 0$.
Let $z = 0^{2i+1}$.
Then $xz = 0^{i^2+2+i+1} = 0^{(i+1)^2} \in L$.
On the other hand, $yz = 0^{i^2+2j+1} \notin L$, because $i^2 < i^2+2j+1 < (i+1)^2$.
Thus, $z$ distinguishes $x$ and $y$.
We conclude that $L$ is an infinite fooling set for $L$, so $L$ cannot be regular.

Solution:
Let $x$ and $y$ be distinct arbitrary strings in $0^*$. Without loss of generality, $x = 0^i$ and $y = 0^j$ for some $i > j \geq 0$.
Let $z = 0^{i^2+i+1}$.
Then $xz = 0^{i^2+2+i+1} = 0^{(i+1)^2} \in L$.
On the other hand, $yz = 0^{i^2+i+j+1} \notin L$, because $i^2 < i^2+i+j+1 < (i+1)^2$.
Thus, $z$ distinguishes $x$ and $y$.
We conclude that $0^*$ is an infinite fooling set for $L$, so $L$ cannot be regular.

Solution:
Let $x$ and $y$ be distinct arbitrary strings in $0000^*$. Without loss of generality, $x = 0^i$ and $y = 0^j$ for some $i > j \geq 3$.
Let $z = 0^{i^2-i}$.
Then $xz = 0^{i^2} \in L$.
On the other hand, $yz = 0^{i^2-i+j} \notin L$, because
\[ (i-1)^2 = i^2 - 2i + 1 < i^2 - i < i^2 - i + j < i^2. \]
(The first inequalities requires $i \geq 2$, and the second $j \geq 1$.)
Thus, $z$ distinguishes $x$ and $y$.
We conclude that $0000^*$ is an infinite fooling set for $L$, so $L$ cannot be regular.
\{w \in (0 + 1)^* \mid w \text{ is the binary representation of a perfect square}\}

**Solution:**

We design our fooling set around numbers of the form \((2^k + 1)^2 = 2^{2k} + 2^{k+1} + 1 = 10^{k-2}10^k1 \in L\), for any integer \(k \geq 2\). The argument is somewhat simpler if we further restrict \(k\) to be even.

Let \(F = 1(00)^*1\), and let \(x\) and \(y\) be arbitrary strings in \(F\).

Then \(x = 10^{2i-2}1\) and \(y = 10^{2j-2}1\), for some positive integers \(i \neq j\).

Without loss of generality, assume \(i < j\). (Otherwise, swap \(x\) and \(y\).)

Let \(z = 0^{2i}1\).

Then \(xz = 10^{2i-2}10^{2i}1\) is the binary representation of \(2^{4i} + 2^{2i+1} + 1 = (2^{2i} + 1)^2\), and therefore \(xz \in L\).

On the other hand, \(yz = 10^{2j-2}10^{2i}1\) is the binary representation of \(2^{2i+2j} + 2^{2i+1} + 1\). Simple algebra gives us the inequalities

\[
(2^{i+j})^2 = 2^{2i+2j} < 2^{2i+2j} + 2^{2i+1} + 1 < 2^{2(i+j)} + 2^{i+j+1} + 1 = (2^{i+j} + 1)^2.
\]

So \(2^{2i+2j} + 2^{2i+1} + 1\) lies between two consecutive perfect squares, and thus is not a perfect square, which implies that \(yz \notin L\).

We conclude that \(F\) is a fooling set for \(L\). Because \(F\) is infinite, \(L\) cannot be regular.