Describe a DFA that accepts the following language over the alphabet $\Sigma = \{0, 1\}$.

1. DFA for all strings in which the number of 0s is even and the number of 1s is not divisible by 3.

Solution:

We use a standard product construction of two DFAs, one accepting strings with an even number of 0s, and the other accepting strings where the number of 1s is not a multiple of 3. The product DFA has six states, each labeled with a pair of integers, one indicating the number 0s read modulo 2, the other indicating the number of 1s read modulo 3.

\[
Q := \{0, 1\} \times \{0, 1, 2\} \\
\mathcal{A} := \{(0, 1), (0, 2)\} \\
\delta((q, r), 0) := (q + 1 \mod 2, r) \\
\delta((q, r), 1) := (q, r + 1 \mod 3)
\]

In this case, the product DFA is simple enough that we can just draw it out in full. I have drawn the two factor DFAs (in gray) to the left and above for reference.

2. DFA for all strings that are both the binary representation of an integer divisible by 3 and the ternary (base-3) representation of an integer divisible by 4.

For example, the string 1100 is an element of this language, because it represents $2^3 + 2^2 = 12$ in binary and $3^3 + 3^2 = 36$ in ternary.

Solution:

Again, we use a standard product construction of two DFAs, one accepting binary strings divisible by 3, the other accepting ternary strings divisible by 4. The product DFA has twelve states, each labeled
with a pair of integers: The binary value read so far modulo 3, and the ternary value read so far modulo 4.

\[ \textbf{Q} := \{0,1,2\} \times \{0,1,2,3\} \]
\[ s := (0,0) \]
\[ A := \{(0,0)\} \]
\[ \delta((q,r),0) := (2q \mod 3, \quad 3r \mod 4) \]
\[ \delta((q,r),1) := (2q + 1 \mod 3, \quad 3r + 1 \mod 4) \]

For reference, here is a drawing of the DFA, with the two factor DFAs (in gray) to the left and above; we would not expect you to draw this, especially on exams. More importantly we would expect you \textbf{not} to draw this, \textbf{especially} on exams. The states of the factor DFA that maintains ternary-value-mod-4 are deliberately “out of order” to simplify the drawing.

For all strings \( w \) such that \( \left( \left\lfloor \frac{|w|}{2} \right\rfloor \right) \mod 6 = 4 \). (\textbf{Hint:} Maintain both \( \left( \left\lfloor \frac{|w|}{2} \right\rfloor \right) \mod 6 \) and \( |w| \mod 6 \).)

\textbf{Solution:}

Our DFA has 36 states, each labeled with a pair of integers representing \( \left( \left\lfloor \frac{|x|}{2} \right\rfloor \right) \mod 6 \) and \( |x| \mod 6 \), where \( x \) is the prefix of the input read so far.

\[ \textbf{Q} := \{0,1,2,3,4,5\} \times \{0,1,2,3,4,5\} \]
\[ s := \{(0,0)\} \]
\[ A := \{(4,r) \mid r \in \{0,1,2,3,4,5\}\} \]
\[ \delta((q,r),0) := (q + r \mod 6, \quad r + 1 \mod 6) \]
\[ \delta((q,r),1) := (q + r \mod 6, \quad r + 1 \mod 6) \]

The transition function exploits the identity \( \binom{n+1}{2} = \binom{n}{2} + n \).
Solution:
The language is identical to the set of strings $w$ such that $|w| \mod 12 \in \{4, 7\}$. This language can be accepted using a 12-state DFA.

(Hard.) All strings $w$ such that $F_{\#(10, w)} \mod 10 = 4$, where $\#(10, w)$ denotes the number of times 10 appears as a substring of $w$, and $F_n$ is the $n$th Fibonacci number:

$$F_n = \begin{cases} 
0 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
F_{n-1} + F_{n-2} & \text{otherwise}
\end{cases}$$

Solution:
Our DFA has 200 states, each labeled with three values:

- $F_k$ mod 10, where $k$ is the number of times we have seen the substring 10.
- $F_{k+1}$ mod 10, where $k$ is the number of times we have seen the substring 10.
- The last symbol read (or 0 if we have read nothing yet)

Here is the formal description:

$$Q := \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \times \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \times \{0, 1\}$$

$$s := \{(0,1,0)\}$$

$$A := \{(4,r,a) \mid r \in \{0,1,2,3,4,5,6,7,8,9\} \text{ and } a \in \{0,1\}\}$$

$$\delta((q,r,0),0) := (q,r,0)$$
$$\delta((q,r,1),0) := (r,q+r \mod 10,0)$$
$$\delta((q,r,0),1) := (q,r,1)$$
$$\delta((q,r,1),1) := (q,r,1)$$

The transition function exploits the recursive definition $F_{k+1} = F_k + F_{k-1}$.

Solution:
The Fibonacci numbers modulo 10 define a repeating sequence with period 60. So this language can be accepted by a DFA with “only” 120 states.